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The Kinetic Energy Spectrum for Turbulence in a Stably Stratified Fluid : Kolmogorov or The Elusive Bolgiano-Obukhov?

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[Abstract : In a homogeneous isotropic fluid the kinetic energy spectrum is supposed to follow the Kolmogorov law. This fact has been very clearly established both experimentally and numerically. More than sixty years ago it was predicted independently by Bolgiano and by Obukhov that for a stratified fluid (like our atmosphere which supports a temperature gradient), the kinetic energy spectrum should be different. The degree of stratification is determined by the Richardson number Ri which is a ratio of the "stratification potential energy" to the kinetic energy. It would be "natural" to find the Bolgiano-Obukhov spectrum at a relatively high Richardson number. However, till now this spectrum has never been clearly seen.

In this article we introduce and discuss the energy spectrum for fully developed turbulence and try to provide the reason behind the inability to capture

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the Bolgiano-Obukhov spectrum. We use an analysis based on an almost forgotten Heisenberg- Chandrasekhar picture of turbulence to establish the crossover function for Kolmogorov to Bolgiano-Obukhov scaling in fully developed turbulence in a stably stratified fluid. We find that there are actually two crossovers-one from the Kolmogorov to the Bolgiano-Obukhov form and the other from the Bolgiano-Obukhov to a pre-dissipative form. For a given Richardson number, the former happens at a wave-number proportional to $Ri^{-3/4}$ and the latter at a wave-number proportional to $Ri^{-3/16}$. This severely restricts the range over which a pure Bolgiano-Obukhov scaling can be seen and explains the elusive nature of that scaling law].

1. Introduction to Turbulent Flows

One of the striking results in the theory of fully developed turbulence in a homogeneous fluid is the scaling law for the energy spectrum $E(k)$ at a given wave-number k in the inertial range. The energy spectrum is related to the total kinetic energy K as

$$K = \frac{1}{2V}\int \left\langle u^2 \right\rangle d^3r = \frac{1}{2}\int \left\langle u_j(k)u_j(-k)\right\rangle \frac{d^3k}{(2\pi)^3} = \int_0^\infty E(k)dk \quad \ldots (1.1)$$

In the above $u_j(\vec{r},t)$ is the random turbulent velocity field, $u_j(k)$ its Fourier transform, the angular bracket denotes an appropriate ensemble average and V is the total volume of the fluid. The inertial range is the range of spatial scales where the wave-number k (inverse of the spatial scale) is smaller than wave-numbers in the dissipation range (where viscous forces dominate) and larger than those corresponding to the large length scales where energy is injected in the fluid to maintain a non-equilibrium steady state. The scales are easily visualized from the stirring of a cup of tea to mix the sugar. We stir the liquid at a scale which is characterized by the radius of the cup (a few centimetres). This energy is dissipated by the viscosity of the liquid which operates at a sub-micron scale. The ratio of the scales is about five orders of magnitude. This is where Kolmogorov[1] argued that the energy spectrum has the universal form $E(k) \propto k^{-5/3}$.

To get a feel for the logic behind the Kolmogorov result , we write down the primary equation for three dimensional incompressible fluid flow with velocity $u_i (i = 1,2,3)$ [incompressibility implies $\partial_i u_i = 0$ (divergence free flow)]

$$\dot{u}_i + u_j \partial_j u_i = -\partial_i p + \nu \nabla^2 u_i + f_i \qquad \dots \; (1.2)$$

In the above equation, the pressure is denoted by ‘p’ (it is actually the pressure per unit density i.e. the constant density of the incompressible fluid has been absorbed in the pressure), $\vec{f}$ is an external force and ν is the kinematic viscosity. To find the rate of change of the total kinetic energy of the fluid, we need to consider the quantity $\int d^3 r \dot{u}_j u_j$ and use Eq.(1.2) to substitute for $\dot{u}_j$.

We will get four terms. The equality $\int dV u_j \partial_j p = \int dV \partial_j (u_j p)$ holds since the velocity field is divergence free and using Gauss’s law this integral becomes the surface integral of the vector $p\vec{u}$ with the surface as far away as we want. In particular the surface can be chosen where the fluid velocity vanishes (there is no source) and hence the integral is zero. Similarly for the term $u_i (u_j \partial_j u_i)$, we can write it as $\partial_j (u_j u^2 / 2)$ and the logic of the previous sentence makes the integral of this term zero as well. The integral $\int u_j \nabla^2 u_j dV$ can be written as the integral of the negative of $\left(\partial_j u_i\right)^2$ and is the energy loss due to the viscous action. On the other hand, $\int dV(u_j f_j)$ is the rate of energy supply by the external force [the stirring action for the example of the coffee cup] . If these two effects are equal in magnitude then the total kinetic energy of the fluid is maintained and we have a non-equilibrium steady state of maintained turbulence. The supply of energy is at large length scales and the dissipation is at the smallest scales. The energy which is introduced at a constant rate ε is pictured to cascade down the scales without loss to be dissipated by viscous action at smallest scales.

Kolmogorov argued that if one is not too close to the dissipation scale and also significantly away from the energy input scale, then the energy spectrum $E(k)$ of turbulence is determined by only the wave-number 'k' and the strength ε of the cascading energy. The dimension of $E(k)$ is L^3/T^2 and that of ε is L^2/T^3. Since $E(k)$ is determined by ε and k, we write $E(k) = C\varepsilon^m k^n$ (the self-similar nature of the turbulent flow made famous by the Leonardo da Vinci drawing), where C is a dimension-free universal constant and the exponents need to be found by a dimensional analysis. We see that $L^3/T^2 = L^{2m-n}/T^{3m}$, leading to $m = 2/3$ and $n = -5/3$. Hence $E(k) = C\varepsilon^{2/3}k^{-5/3}$, the so-called Kolmogorov spectrum. This is one of the best known results in all of turbulence and generally known as Kolmogorov's 5/3 law.

We now shift our attention to the stably stratified fluid where the density decreases with height. The stratified fluid was first studied by

Bolgiano[2] and Obukhov[3], who argued that one is supposed to encounter an energy spectrum $E(k)$ which scales as $k^{-11/5}$ in this case. We will work with a positive temperature gradient in the z-direction which makes the stratification stable. The constant positive temperature gradient is $\Delta T / d$ with the bounding surfaces as the x-y planes located at $z = 0$ and $z = d$. The steady state temperature profile is linear in z. We work in the Boussinesque approximation[4] where the buoyancy induced temperature fluctuation around the steady state shows up only in the linear order in the velocity dynamics. We denote the dimensionless temperature fluctuation by $\theta = \delta T / \Delta T$. We also include a random forcing term $\vec{f}(\vec{r},t)$ in the velocity dynamics to inject energy at large length scales. With the buoyancy force included, the velocity dynamics (Navier-Stokes equation) becomes

$$\partial_t u_i + \left(u_j \partial_j\right) u_i = -\frac{\partial_i p}{\rho_0} - \alpha \Delta T g \theta \hat{z} + \nu \nabla^2 u_i + f_i \quad \ldots \quad (1.2)$$

The pressure field is denoted by $p(\vec{r},t)$, ρ_0 is a mean density (will be absorbed in the pressure subsequently), α is the expansion coefficient and ν the kinematic viscosity. The dynamics of θ for stable stratification (including an external random fluctuation $h(\vec{r},t)$ at large length scales) is

$$\partial_t \theta + u_j \partial_j \theta = \lambda \nabla^2 \theta + \frac{u_z}{d} + h(\vec{r},t) \quad \ldots (1.3)$$

The dynamics of the total kinetic energy K follows from Eq. (1.2) as

$$V \partial_t K = \alpha \Delta T g \int \theta u_z d^3 r - \nu \int \left(\partial_i u_j\right)^2 d^3 r + \int f_j u_j d^3 r \quad \ldots (1.4)$$

The first term on the right hand side ensures that K is not conserved in the unforced, inviscid limit. In a similar vein, from Eq. (1.3) we have

$$\partial_t \frac{1}{V}\int \frac{1}{2}\theta^2 d^3r = \frac{1}{V}[\frac{1}{d}\int \theta u_z d^3r - \lambda\int\left(\nabla\theta\right)^2 d^3r + \int \theta h d^3r] \quad \text{… (1.5)}$$

Between Eqns. (1.4) and (1.5), we have a conserved quantity E in the unforced and dissipation regime which is like a sum of kinetic and potential energies[5], since

$$\partial_t E = \partial_t (K + \alpha\Delta Tgd \frac{1}{V}\int \frac{1}{2}\theta^2 d^3r)$$

$$= -\frac{1}{V}[\nu\int\left(\partial_i u_j\right)^2 d^3r + \lambda\alpha\Delta Tgd\int\left(\nabla\theta\right)^2 d^3r - \int (f_j u_j + \theta h) d^3r] \quad \text{…(1.6)}$$

The first two terms on the right hand side cause dissipation at very short length scales and the third and fourth terms inject energy at large length scales. In the unforced and inviscid limit ($\nu=\lambda=f=h=0$) the quantity $E = K + (\alpha\Delta Tgd)U$ where $U = \frac{1}{2V}\int \theta^2 d^3r$ is conserved and from the structure of Eq (1.6), this quantity E is produced at large length scales (small wave-number scales) and flows down to short length scales (large wave-numbers) where it is dissipated. To compare K and U on the same footing it is best to make them have the same dimension and this is done by defining $\bar{U} = u_0^2 U$ where u_0^2 is a mean square velocity. The conserved quantity in the inviscid, unforced limit is now $K + \frac{\alpha\Delta Tgd}{u_0^2}\bar{U} = K + Ri\bar{U}$ where the dimensionless number

$Ri = \frac{\alpha \Delta T g d}{u_0^2}$ is called the Richardson number. For the stably stratified fluid, this is the energy that is conserved in the absence of dissipation and external forcing. However, the energy spectrum of turbulence that one talks about is always the kinetic energy spectrum. For the energy flux, however, it is the total energy and that can lead to a very different story for the scaling laws[6,7]. It should be noted that for the convective situation (top-heavy) dealt with in Refs[8-16], the terms which are quadratic in the θ-field in Eq.(1.6) appear with a negative sign. This makes definite statements about the sign of the energy flux difficult as the flux may depend on the value of the Prandtl number. A detailed discussion can be found in Verma et al[17].

When the Richardson number becomes high, the $\bar{U}$ term can dominate the "energy" transfer and the energy spectrum will be changed because the transfer will now be engineered by the θ^2 term. The rate of transfer ε_θ will have the dimension of θ^2/t where t is time. Once again, it is important to appreciate what happens at large Richardson numbers. The scale to scale transfer of the energy E is now dominated by the dynamics of the temperature fluctuation $\theta(\mathbf{r},t)$. Apparently θ is "dimensionless" but this is not the dimension one is talking about. The dimension is the scaling dimension and corresponds to the dimension that one gets if Eq. (1.3) is going to be invariant under a scale transformation. This implies that the constant rate ε_θ at which the energy is transferred in the inertial range will have the dimension L^2/T^5. The energy spectrum $E(k)$ in this limit will be determined by ε_θ and k and is easily seen to be

$$E(k) = K_1 \varepsilon_\theta^{2/5} k^{-11/5} \qquad \ldots (1.7)$$

where K_1 is a numerical constant. The above spectrum is known as the Bolgiano-Obukhov scaling law[23]. Unlike the Kolmogorov spectrum, this spectrum has hardly ever been observed. Two important exceptions are the investigations of Kumar et al[6] and Rosenberg et al[7]. Even in these two studies, the 11/5 spectrum is seen over only one decade at the most.

The scaling that we describe in Eq. (1.7) is isotropic while the situation that we have described is quite clearly anisotropic. Many of the references[18-23] do observe an anisotropic spectrum. This is why this issue was studied from a scaling perspective in Refs[24-25] and it was found that the isotropic Bolgiano-Obukhov spectrum would be a reasonable approximation when the Richardson number is of $O(1)$ and the vertical length scale is of $O(u_0^3 / \varepsilon)$ which is in agreement with the finding of Rosenberg et al[7]. In this moderately anisotropic situation, Eq.(1.7) has to be understood as an angle averaged result[26].

In a recent work[27], we suggested, based on a preliminary examination of the local energy transfer associated with Eq.(1.6), that the Bolgiano –Obukhov scaling should be seen at wave-numbers higher than those at which the Kolmogorov scaling is seen. This is anti-'common-sensical' since for large k, we will have $k^{-5/3} > k^{-11/5}$ and hence Kolmogorov spectrum should dominate. However, the 'common-sensical' result has never been seen. In fact, an examination of the data presented by Rosenberg et al[7] was actually seen to be consistent with the violation of naive reasoning. In fact, for convective turbulence a similar qualitative behaviour in co-ordinate space was seen by Kunnen et al[15].

Our next goal is to discuss the intricacies associated with the scaling of the energy spectrum $E(k)$ for the stably stratified fluid. The major issues (which have prevented a clear cut observation of the Bolgiano-Obukhov spectrum) are:

(A) The crossover from Kolmogorov spectrum to Bolgiano-Obukhov spectrum is determined by the combination $kRi^{3/4}$ with the Kolmogorov region corresponding to $kRi^{3/4} << 1$ and the Bolgiano-Obukhov region corresponding to $kRi^{3/4} > 1$. This means that even if one is at a reasonably high Richardson number, one could be seeing a Kolmogorov spectrum if the condition $kRi^{3/4} > 1$ is not satisfied. What is very likely, even if it is , one will be caught in a crossover region where the exponent will seem to lie between 1.67 and 2.2. The crossover region is consequently vital and we will obtain an exact differential equation describing the course of it.

(B) The problem gets further complicated by the fact that the Bolgiano-Obukhov spectrum crosses over to an intermediate scaling in the pre-dissipative regime. This crossover happens if $kRi^{3/16} > 1$ and hence if the Richardson number is not ideally chosen, a clear run of the exponent 2.2 would hardly be seen. We will provide an exact form for this crossover as well. Between these two crossovers , it becomes non-trivial to see a pure Bolgiano-Obukhov spectrum and this could be the reason that , unlike the Kolmogorov spectrum, there are very few instances of finding a pure Bolgiano-Obukhov spectrum.

In Sec II, we extend the previous works of Chandrasekhar[28] and Heisenberg[29,30] to study the crossover from the Kolmogorov to a dissipative regime. We show how the Heisenberg formulation works for the moderately high Richardson number case (energy transfer by the thermal fluctuations only) and obtain a closed-form expression for the spectrum describing the transition from a Bolgiani-Obukhov form to a pre-dissipative form. In Sec III we generalize this approach to obtain a gradual crossover from the Kolmogorov spectrum to the Bolgiano-Obukhov one. We conclude with a brief summary in Sec IV.

2 Bolgiano-Obukhov to dissipation range crossover

In this section we extend the Heisenberg –Chandrasekhar[28-30] formulation to the large Richardson number situation where the energy dynamics of Eq. (1.7) contains a significant contribution from thermal fluctuations and the dynamics is primarily the dynamics of the θ-field as given by Eq.(1.3). In Fourier space the θ-dynamics is given by

$$\dot{\theta}(p) = -i\sqrt{V}\int p_j u_j(\vec{p}-\vec{q})\theta(q)\frac{d^3q}{(2\pi)^3} - \lambda p^2\theta(p) + \frac{u_3}{d} \qquad \ldots (2.1)$$

The total energy E must include the "potential energy" term in Eq. (1.12) and in Fourier space is written as

$$E = K + u_0^2 Ri\int \frac{d^3p}{2(2\pi)^3}\left\langle \theta(p)\theta^*(p)\right\rangle \qquad \ldots (2.2)$$

Dropping all constant pre-factors, we write this as $E = K + Ri\int d^3pF(p)$, where $F(p) = \left\langle \theta(p)\theta^*(p)\right\rangle$ and is the amount of "potential" energy at the scale p. It should be noted that the last term on the right hand side of Eq.(2.1) is like the first term with the momentum $p \sim d^{-1} << 1$ and hence has been ignored. In the large Richardson number situation it is the time derivative of E at a given scale which is dominated by $\dot{F}(p)$. The energy spectrum that one talks about is, however, always the kinetic energy spectrum unless specifically mentioned otherwise.

The dynamics of the potential energy is the primary contributor to the energy flux at moderate Richardson numbers. The dynamics of $F(p)$ is (cancelling the ubiquitous Ri in this limit)

$$\dot{F}(p) = -\mathrm{Im}\sqrt{V}\theta^*(p)\int p_j u_j(p-q)\theta(q)\frac{d^3q}{(2\pi)^3} - 2\lambda p^2 F(p) \qquad \ldots (2.3)$$

$$= T_\theta(p) - 2\lambda p^2 F(p)$$

In the above the first term is the transfer due to the interacting triad, $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}-\boldsymbol{q}$. Once again the total energy $\bar{E}(k)$ contained between the scales p=0 and p=k is obtained as $\int_0^k 4\pi p^2 F(p)dp$ and the time derivative of $\bar{E}(k)$ is the rate at which energy is leaving the region $p<k$ for the region $p>k$ and hence is the rate of energy transfer from wave-numbers below k to those above it. Hence the transfer rate, dominated by the second term in Eq.(2.2), is found as (the constant velocity scale u_0^2 is absorbed in Ri)

$$\varepsilon_\theta(k) = Ri[\int_0^k 4\pi T_\theta(p)p^2 dp - \lambda\int_0^k 4\pi p^4 F(p)dp] \qquad \ldots (2.4)$$

We first take the dissipative term in the above equation and express it in terms of the energy spectrum $E(p)$ and p. The dimension of $F(p)$ is found from the dimension of $\theta(p)$. As explained before the dimension that we are talking about is the scaling dimension (how do equations remain invariant under a scale transformation) and hence the scaling dimension of $\theta(r)$ is L/T^2 . Consequently, $F(p)$ has a scaling dimension of L^5/T^4. As a result in the second term on the right hand side of Eq. (2.4), the quantity $p^4F(p)$ has the dimension L/T^4. Expressed in terms of p and $E(p)$ it behaves as $E^2(p)p^5$. The second term on the right hand side of Eq. (2.4) now becomes (dropping numerical factors) $\int_0^k E^2(p)p^5dp$. Our task now is to cast the first term in a similar form i.e. we want to write it as $-2\lambda_{eff}\int_0^k E^2(p)p^5dp$. Once again as in the kinetic energy case, the λ_{eff} operates at all scales that are greater than k and is better written as $\int_k^\infty \overline{\lambda}_{eff}(p)dp$. As in the Kolmogorov case[28] expressing $\lambda_{eff}(p)$ (this is done simply by a dimensional analysis) in terms of $E(p)$ and p, we get (the sign has been made positive with the understanding that the flow is from low to high values of k)

$$\varepsilon_\theta(k) = Ri\left[\int_k^\infty \sqrt{\frac{E(p)}{p}}\frac{dp}{p} + \lambda\right]\int_0^k E^2(p)p^5dp \qquad \dots (2.5)$$

With $\varepsilon_\theta(k)$ set equal to a constant ε we obtain the crossover from the inertial range scaling to a pre-dissipative scaling for Bolgiano-Obukhov turbulence.

Taking a derivative of Eq. (2.5) with respect to k when $\varepsilon_\theta(k)$ is a constant, gives

$$\left[\int_k^\infty \sqrt{\frac{E(p)}{p^{3/2}}}dp+\lambda\right]E^2(k)k^5=\sqrt{\frac{E(k)}{k^3}}\int_0^k E^2(p)p^5dp \qquad \dots (2.6)$$

Defining $y(k)=\int_0^k E^2(p)p^5dp$, we can write Eq. (2.6) as

$$\lambda+\int_k^\infty \sqrt{\frac{E(p)}{p^{3/2}}}dp=\frac{1}{k^2(k^3E(k))^{3/2}}y(k) \qquad \dots (2.7)$$

Further, defining $\sqrt{E(p)p^{3/2}}$ as $g^{1/4}$, the above equation becomes

$$\lambda+\int_k^\infty \frac{g(p)^{1/4}}{p^3}dp-\frac{y(k)}{k^2g(k)^{3/4}}=0 \qquad \dots (2.8)$$

The definition of $y(k)$ shows $\frac{dy}{dk}=k^5E^2(k)=\frac{g(k)}{k}$ and using this in Eq. (2.8) to change the variable *p* to *y*, we arrive at

$$\lambda+\int_{y(k)}^\infty \frac{dy}{k^2g^{3/4}}-\frac{1}{k^2}\frac{y(k)}{g^{3/4}}=0 \qquad \dots (2.9)$$

Differentiating the above equation with respect to *y* yields (note that

$$\frac{d(k^2)}{dy}=\frac{2k^2}{g(k)}-\frac{1}{g^{3/4}}+\frac{2y}{g^{7/4}}-\frac{d}{dy}\left(\frac{y}{g^{3/4}}\right)=0 \qquad \dots (2.10)$$

leading to the differential equation $\frac{dg}{dy}-\frac{8g}{3y}=-\frac{8}{3}$ with the solution $g(y)=\frac{8y}{5}-Ay^{8/3}$ where A is a constant. The relation $\frac{dy}{dk}=\frac{g}{k}$ allows us to write

$$\ln k = \int \frac{dy}{g(y)} + \text{const.} \qquad \ldots (2.11)$$

Integrating we find

$$y = \frac{\beta k^{8/5}}{(1+\alpha k^{8/3})^{3/5}} \qquad \ldots (2.12)$$

where α and β are constants. We use the definition of $y(k)$ to write

$$E^2(k)k^5 = \frac{dy}{dk} = \frac{8\beta}{5}\frac{k^{3/5}}{\left(1+\alpha k^{8/3}\right)^{8/5}} \qquad \ldots (2.13)$$

In the low k inertial range, the k term in the denominator is unimportant and we get

$$E(k) \propto k^{-11/5} \qquad (2.14)$$

-the desired Bolgiano spectrum. For high values of k the crossover is to a $E(k) \propto k^{-13/3}$ form, which is less steep than the Kolmogorov to viscous crossover.

We now need to discuss the Richardson number dependence of the coefficients α and β in Eq.(2.13) above. For this, we need to go back to Eq.(1.7) and note that $E(k) \propto {\varepsilon_\theta}^{2/5}$ and using Eq. (2.4), we get $E(k) \propto Ri^{2/5}$. Using Eq.(2.13) in the inertial range we get $\beta \propto Ri^{4/5}$. In the high k range, dissipation plays a more important role and hence the spectrum there will be determined by the dissipation coefficient λ and not Ri. In the high wave-number range the spectrum will be independent of Ri if $\alpha \propto \sqrt{Ri}$. This implies that the spectrum will be proportional to

$k^{-13/3}$ for wave-numbers $k >> Ri^{-3/16}$. This is our first result, and it states that to see the Bolgiano- Obukhov spectrum, we need to focus on wave-numbers which satisfy $kRi^{3/16} < 1$. We define a crossover wave-number $k_C = Ri^{-3/16}$. For $k < k_C$, one has the Bolgiano –Obukhov spectrum and for $k > k_C$, one enters the dissipation range with a $k^{-13/3}$ spectrum. Since the coefficient α in Eq.(2.13) is proportional to $\sqrt{Ri}$, we can write Eq.(2.13) as

$$E(k) = \bar{E} \frac{(k / k_C)^{-11/5}}{\left[1 + \left(\frac{k}{k_C}\right)^{8/3}\right]^{4/5}} \qquad \ldots (2.15)$$

The above formula clearly shows the transition from the Bolgano-Obukhov spectrum to a dissipation influenced spectrum as the wave-number increases past the crossover wave-number k_C. In the next section, we present the Kolmogorov to Bolgiano-Obukhov spectrum which will yield another constraint on the range where one can see the stratified fluid spectrum. It should be clear that our calculation does not fix the coefficient of the Ri involving terms. So the exact range where the desired spectrum will be seen is not being set down but a clear idea of where to look for it and the high probability of being in a crossover range forever are the two points that we want to bring out.

3. *The Kolmogorov to Bolgiano-Obukhov crossover*

In this section we use the technique developed above to obtain the crossover from the Kolmogorov to the Bolgiano-Obukhov spectrum. A preliminary version of this can be found in Ref [27]. We begin by noting that the time derivative of the $E(k)$ obtained from the full energy expression in the first line of Eq. (1.5) gives an energy flux which is simply the sum of the kinetic energy flux and the "potential energy" flux of the previous section with the latter weighted by the appropriate factor of $\alpha \Delta T g d$. In this section we will focus on the inertial range crossover only and hence drop the dissipative terms. The total energy flux $\varepsilon_T(k)$ across the wave-number k is given by the appropriate combination of $\dot{C}(p)$ and $\dot{F}(p)$ where $C(p) = \langle u_\alpha(p) u_\alpha(-p) \rangle$ is the velocity correlation function. In analogy with the previous section the rate of total energy transfer from wave-numbers below 'k' to wave-numbers above 'k' is

$$\varepsilon_T(k) = \int_0^k 4\pi p^2 \left[\dot{C}(p) + Ri \dot{F}(p) \right] \qquad \ldots (3.1)$$

Since we are in the inertial range the dissipative terms will be dropped and we have the energy transfer rate given by

$$\varepsilon_T(k) = \int_k^\infty \frac{dp}{p} \sqrt{\frac{E(p)}{p}} \left[\int_0^k p^2 E(p) dp + Ri \int_0^k p^5 E^2(p) dp \right] \qquad \ldots (3.2)$$

The first term on the right hand side of the above equation is exactly the term used by Heisenberg [31]. We need to point out that being in the inertial range puts a limit on the wave number. The smallest allowed wave number is determined by the inverse of the system size. The largest requires that $\sqrt{E(k)/k}$ be significantly larger than ν or λ, whichever is larger. This requires $\varepsilon^{1/3} k^{-4/3} >> \nu, \lambda$ as well as $Ri^{1/5} k^{-8/5} >> \nu, \lambda$. It should be pointed out that we have absorbed a constant dimensional factor of square of a typical velocity scale in the problem in the Richardson number written above. We define

$$y(k) = \int_0^k p^2 E(p)dp + Ri\int_0^k p^5 E(p)dp \qquad \text{… (3.3)}$$

$$g(k) = k^3 E(k) + Rik^6 E^2(k) \qquad \text{… (3.4)}$$

$$\frac{dy}{dk} = \frac{g(k)}{k} \qquad \text{… (3.5)}$$

Invoking the Kolmogorov picture of the inertial range where $\varepsilon_T(k)$ is a constant and taking a derivative of Eq. (3.2) with respect to *k*, we obtain (it should be noted that since the arguments leading to Eq. (3.2) from Eq.(3.1) are based on dimensional analysis, there can be unknown functions of the dimensionless variable Ri associated with the second term in Eq. (3.4) which can only be fixed by some additional constraint)

$$y(k)\sqrt{\frac{E(k)}{k^3}} = \frac{dy}{dk}\int_k^\infty \sqrt{\frac{E(p)}{p}}\frac{dp}{p} \qquad \text{… (3.6)}$$

Using Eq.(3.3) to express dy/dk, we get

$$\int_k^\infty \sqrt{\frac{E(p)}{p}}\frac{dp}{p} = \sqrt{\frac{E(k)}{k^3}}\frac{y(k)}{k^2 E(k) + Rik^5 E^2(k)}$$
$$= \frac{y(k)}{k^2}\frac{1}{\left(k^3 E(k)\right)^{1/2}}\frac{1}{1 + Rik^3 E(k)} \qquad \text{… (3.7)}$$

Solving the quadratic equation for $k^3E(k)$ in Eq. (3.4), one has

$$E(k)k^3 = \frac{2g(k)}{[1+4Rig(k)]^{1/2}+1} \qquad \dots (3.8)$$

Substituting this result in Eq. (3.7) allows us to write the left hand side of the equation as

$$\int_k^\infty \sqrt{\frac{E(p)}{p^3}}dp = \int_k^\infty \frac{dp}{p^3}\left[\frac{\sqrt{1+4Rig(p)}-1}{2Ri}\right]^{1/2}$$

$$= \int_{y(k)}^\infty \frac{dy}{p^2\sqrt{g(p)}}\left[\frac{2}{1+\sqrt{1+4Rig(p)}}\right] \qquad \dots (3.9)$$

We have used Eq. (3.5) to obtain the final form above.

Using Eq.(3.8) in the right hand side of Eq.(3.7) and noting that

$$\frac{1}{\sqrt{k^3E(k)}}\frac{1}{1+Rik^3E(k)} = \frac{y(k)}{k^2\sqrt{g(k)}}\left[\frac{2}{1+\sqrt{1+4Rig(k)}}\right]^{1/2} \qquad \dots (3.10)$$

We finally write Eq.(3.7) in the form

$$\int_{y(k)}^\infty \frac{dy}{g(p)p^2}\left[\sqrt{1+4Rig(p)}-1\right]^{1/2} = \frac{y(k)}{k^2g(k)}\left[\sqrt{1+4Rig(k)}-1\right]^{1/2} \qquad \dots (3.14)$$

Differentiating both sides of Eq (3.11) with respect to *y*, we have

$$\frac{\left(\sqrt{1+4gRi}-1\right)^{1/2}}{g(k)} = \frac{2y(k)}{g^2}\left(\sqrt{1+4Rig}-1\right)^{1/2}$$

$$-\frac{d}{dy}\left(\frac{y\left[\sqrt{1+4Rig}-1\right]^{1/2}}{g}\right) \qquad \dots (3.12)$$

Defining,

$$f(y) = \frac{\left(\sqrt{1+4Rig(k)}-1\right)^{1/2}}{g(k)} \qquad \text{… (3.13)}$$

leads to

$$f = \frac{2y(k)f}{g(k)} - \frac{d}{dy}(yf) \qquad \text{… (3.14)}$$

Consequently,

$$d(\ln fy^2) = 2\frac{dy}{g} \qquad \text{… (3.15)}$$

with the integral $\sqrt{f}\,\frac{y}{k} = C$ (constant) leading to

$$\frac{\left[\sqrt{1+4Rig(k)}-1\right]^{1/4}}{\sqrt{g(k)}}\,\frac{y}{k} = C \qquad \text{… (3.16)}$$

From Eq (3.5), we now have the $g(k)$ given by (setting $C = \sqrt{2}Ri^{1/4}$)

$$\int_0^k \frac{g(p)}{p}\,dp = kg^{1/4}(k)\left[\sqrt{1+4Rig(k)}+1\right]^{1/4} \qquad \text{… (3.17)}$$

A derivative takes us to the exact differential equation satisfied by $g(k)$. We find

$$\frac{dg}{dk} = \frac{4g^{7/4}(k)}{k^2}\,\frac{\left[1+\sqrt{1+4Rig(k)}\right]^{3/4}}{1+\dfrac{1+6Rig(k)}{\sqrt{1+4Rig(k)}}} - \frac{4g(k)}{k}\,\frac{\left[1+\sqrt{1+4Rig(k)}\right]}{1+\dfrac{1+6Rig(k)}{\sqrt{1+4Rig(k)}}}$$

… (3.18)

To obtain the asymptotic answers, we need to study the limits $Ri \rightarrow 0$ (Kolmogorov) and $Ri \rightarrow \infty$ (Bolgiano-Obukhov). In the first case we obtain

$$\frac{dg}{dk} + \frac{4g}{k} = \frac{4g^{7/4}}{k^2} \qquad \ldots \ (3.19)$$

Simply by inspection we can write down the solution as $g(k) \propto k^{4/3}$. From Eq.(3.4) we now get $E(k) \propto k^{-5/3}$ which is the Kolmogorov spectrum. For $Rig(k) >> 1$ on the other hand

$$\left[\frac{dg}{dk} + \frac{8g}{3k}\right] = \frac{2\sqrt{2}}{k^2}\frac{g(k)^{13/8}}{Ri^{1/8}} \qquad \ldots \ (3.20)$$

In this limit inspection yields $g(k) \propto k^{8/5}$ and this in conjunction with Eq.(3.4) in the $Rig(k) >> 1$ limit gives $E(k) \propto k^{-11/5}$ which is the Bolgiano-Obukhov spectrum. It is thus clear that the exact solution for $g(k)$ obtained by numerically integrating Eq.(3.18) starting at some small value of k with an initially prescribed $g(k)$ will evolve differently for different values of the Richardson number and in the extreme cases $Ri \rightarrow 0$ and $Ri\, g(k) >> 1$ yield the two limiting spectra. The departure from the Kolmogorov region occurs if $Rig(k) > 1$ with $g(k) \propto k^{4/3}$ leading to the constraint $Rik^{4/3} > 1$ which implies that it occurs at wave-numbers $k > Ri^{-3/4}$. We define a wave-number k_B by the relation $k_B = Ri^{-3/4}$ and another wave number k_C by the relation $k_C = Ri^{-3/16}$. The combined conclusion of Secs 2 and 3 is that a clear Bolgiano-Obukhov spectrum can only be seen in the span $k_B < k < k_C$.

4. Discussion of the crossovers

In this section, we use the essential features of the crossover issues in Sections 2 and 3 to write down a handy crossover formula which can be used to analyze experimental and numerical data. To this end, we rewrite Eq.(2.23) as

$$E(k) = \frac{E_0 k^{-11/5}}{\left[1+\left(\frac{k}{k_C}\right)^{8/3}\right]^{4/5}} \qquad \ldots (4.1)$$

where the constant E_0 is dependent on the Richardson number. This provides the crossover from the Bolgiano-Obukhov spectrum to an early dissipative range spectrum. To simplify the Kolmogorov to Bolgiano-Obukhov crossover, we return to Eq (3.18) and simplify it by modifying a couple of inessential details. We carry out approximations in the functional forms involving the square roots to cast everything as a function of the square root $\sqrt{1+Rig(k)}$ alone which leaves the two limiting forms ($Ri \ll 1, Ri \gg 1$) unchanged. What this amounts to is that we take Eq.(3.18) and replace the number '6' appearing in two places by the number '4'.This leads us to a much simplified crossover differential equation

$$\frac{dg}{dk} + \frac{4g}{k} = \frac{2^{7/4}}{k^2} \frac{g^{7/4}}{\left[1+Rig(k)\right]^{1/8}} \qquad \ldots (4.2)$$

It is slightly more convenient to work in terms of $h(k) = g^{-3/4}(k)$ and $l = k^{-1}$,which casts Eq.(4.2) in the form

$$\frac{dh}{dl}+\frac{3h}{l}=\frac{\alpha}{(1+Ri/h^{4/3})^{1/8}} \qquad \text{… (4.3)}$$

where $\alpha = 2^{7/4}$. The above form is easily amenable to perturbation theory for small Richardson numbers and to the lowest non-trivial order

$$h(l)=\frac{\alpha l}{4}-\frac{3\alpha}{64}\left(\frac{4}{\alpha}\right)^{4/3}\frac{Ri}{l^{1/3}}+O(Ri^2) \qquad \text{… (4.4)}$$

It is easy to check that for high Richardson numbers the asymptotic form of $h(l)$ is

$$h(l)=\beta l^{6/5} \qquad \text{… (4.5)}$$

where β is a function of the Richardson number which vanishes for $Ri \to \infty$. An approximate formula which follows the above constraints is

$$h(l)=\frac{\alpha l}{4}\left(\frac{1}{1+\frac{Ri}{l^{4/3}}}\right)^{3/20} \qquad \text{… (4.6)}$$

Remembering l=1/k , $g(k)$=$h(k)^{-4/3}$ and $k_B = Ri^{-3/4}$ we get

$$g(k)=\left(\frac{k}{k_B}\right)^{4/3}\left(1+\frac{k^{4/3}}{k_B^{4/3}}\right)^{1/5} \qquad \text{… (4.7)}$$

Now turning to Eq. (3.8) and using the same approximations as explained below Eq. (4.1) , we arrive at the simplest possible crossover as

$$E(k)=K_0\left(\frac{k}{k_B}\right)^{-3}\frac{g(k)}{\sqrt{1+B_0Rig(k)}} \qquad \text{… (4.8)}$$

In the above equation K_0 and B_0 are numerical constants of order unity which should be material independent and hence the above crossover has a universal character. For $Ri << 1$ Eq. (4.8) yields $E(k) \propto g(k)k^{-3}$ and further $g(k) \propto k^{4/3}$ since in this limit $k / k_B << 1$ .The Kolmogorov energy spectrum is obtained for very small Richardson numbers. For $Rik^{4/3} >> 1$, i.e. $k >> k_B$ we crossover to the Bolgiano spectrum. If we want a single formula to represent the crossover to the spectrum for $k >> k_C$, we can combine Eqs (4.8) and (2.25) to write

$$E(k) = K_0 \left(\frac{k}{k_B} \right)^{-3} \frac{g(k)}{\sqrt{1 + Rig(k)}} \frac{1}{\left[1 + \mu \left(\frac{k}{k_B} \right)^{8/3} \right]^{4/5}} \qquad \ldots (4.9)$$

The number μ in the above formula is the ratio $\mu = (k_B / k_C)^{8/3}$ and is expected to be orders of magnitude smaller than unity. The constant B_0 has been set equal to unity which is consistent with the order of accuracy in the approximations in this section. The compensated functions shown in Fig 1 are obtained from Eq.(4.1) with the overall scale-factor set to unity.

The crossovers are now clearly seen. At any given Richardson number *Ri*, for small wave-numbers satisfying $Rik^{4/3} << 1$ ($k << k_B$), one gets $E(k) \propto k^{-5/3}$. As the wave-number increases, it begins to crossover to $E(k) \propto k^{-11/5}$ and for $k >> k_B$, it is predominantly of the Bolgiano-Obukhov variety. If the wave-number is increased further to $k >> k_C$, the crossover to a faster decay is obtained as shown in Eq. (2.25) with

$E(k) \propto k^{-13/3}$ before one enters a completely dissipation dominated regime. A function covering the entire range can be written down as

$$E(k)k^{11/5} = \frac{x^{8/15}\left[1+x^{4/3}\right]^{1/5}}{\left[1+Rix^{4/3}\left(1+x^{4/3}\right)\right]^{1/2}}\left[\frac{1}{1+\mu x^{8/3}}\right]^{4/5} \qquad \ldots(4.10)$$

For very small Richardson numbers, there is hardly any flat region in the compensated spectrum. It is for Richardson number of order unity that about a decade of flat compensated spectrum is obtained. For higher Richardson number , the anisotropy is expected to play a major role. For $Ri = 1$, our formula yields answers very similar to those seen in Fig ( ) of Ref.(6) and Fig (4b) of Ref.(7). Here it is clearly seen that the spectrum crosses over from Kolmogorov to Bolgiano-Obukhov as the wave-number increases and then departs again both in the calculation here and the simulations.

For completeness, we provide the crossover results in co-ordinate space as well. In co-ordinate space one studies the correlation function $S_2(r) = \left\langle [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})]^2 \right\rangle$. The relation between $S_2(r)$ and $E(k)$ is obtained as

$$S_2(r) = 4\int_0^\infty E(k)\left[1 - \frac{\sin kr}{kr}\right]dk \qquad \ldots (3.26)$$

The energy spectrum $E(k)$ is obtained from Eqs. (3.25) and (3.8) and $S_2(r)$ from Eq.(3.26). The crossover features are as follows. For large values of r corresponding to $r \gg Ri^{3/4}$, the spectrum is Kolmogorov i.e $S_2(r) \propto r^{2/3}$, for $r \ll Ri^{3/4}$ (more precisely $Ri^{5/8}$), the spectrum is

Bolgiano-Obukhov i.e. $S_2(r) \propto r^{6/5}$ and for still smaller r (but still not in the dissipative range) it is $S_2(r) \propto r^{10/3}$ as established by Eq (2.23). This implies that if one observes a Bolgiano-Obukhov spectrum at a certain spatial scale at a low Richardson number, it is possible that one will observe a Kolmogorov spectrum at that same spatial scale at a high Richardson number.

5. Conclusion

We have reviewed the turbulence energy spectrum in stratified fluids and looked at the issue of crossover from Kolmogorov to Bolgiano-Obukhov scaling and beyond in the energy spectrum of a stably stratified fluid (when the results are always true) and in a convecting fluid (when our results hold only if the Bolgiano Obukhov spectrum is numerically or experimentally observed). The key observation is that what determines the crossover from one regime to another is the product $k^n Ri$ where n is a number of order unity. For values of $k^n Ri$ greater than order unity, it is the Bolgiano-Obukhov spectrum which is relevant and for lower than unity values the observed spectra should be Kolmogorov like. The value of n is 4/3 in the extreme Kolmogorov limit and increases to 8/5 for larger Richardson numbers. At values of k significantly higher than that required for onset of the Bolgiano spectrum, the energy spectrum crosses over to a $k^{-13/3}$ form. In co-ordinate space the second order structure factor which is the Fourier transform of the energy spectrum scales as $r^{2/3}$ at large distance scales (Kolmogorov) and crosses over to a Bolgiano-Obukhov spectrum ($r^{6/5}$) at shorter scales and a $r^{10/3}$ form at even shorter scales bordering on the dissipative regime. This is seen in Fig (6) of Ref [15] but not mentioned in the article.

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