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Brownian Motion – An Introduction With A Historical Perspective

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[**Abstract:** Brownian motion, after more than two hundred years of its genesis, still remains relevant as everyday we identify more areas in science where diffusion is the basic mechanism. Apart from that, the theoretical framework of Brownian motion lays the foundation of non equilibrium statistical mechanics. With the advent of its quantum version, other vistas have also opened up, specially where one treats a Brownian motion in a bath of parametric oscillators - a model for interacting theories where we can see the connections between noise, fluctuations, decoherence and dissipation etc. In our modest article we review the classical approaches to the Brownian motion and try show the unity in the diversity of different schemes].

Keywords: diffusion, dissipation, noise, fluctuation

1. Introduction

In 1827, Robert Brown, the Scottish botanist, observed rapid, inherent, incessant and zigzag (extremely irregular) motion of pollen grains of Clarkia pulchella in aqueous suspension. Such motions were known to exist among organic molecules and the cause was attributed to the mysterious 'vital force'- a characteristic of all living matter. Brown, used dead pollens and finely ground inorganic dust to show that the motion still persisted and proved through a series of experiments that the motion was *not due to*

- Vital force, *a mysterious force attributed to all living matter at that time*,
- Currents
- Convection rolls
- Evaporation

Though he was unable to pinpoint the exact cause of the motion, Brown gave birth to a problem whose successful resolution would require almost a century.

It is interesting to note that none of the proponents of the kinetic theory published anything about the Brownian motion. It was forgotten by the then scientific community for some time. It was almost after a span of thirty years the problem was picked up again by Jules Regnault in 1858 who described the motion as a result of the absorption of incident light ending up in local heating of the solvent and creating a microscopic convection current. Interestingly, Christian Wiener, in 1863 argued that the motion is due to the internal molecular motion in the fluid but unfortunately he assumed that there were two kinds of molecules involved in the process, the material molecules and the aether molecules. Subsequent experiments showed that the motion was independent of

- the chemical composition or the shape of the container
- External effects (e.g. incident light, wind etc.)

It was during 1874-1880 that the right kind of ideas started emerging. Three Belgian Jesuits, J.Delsaux, J. Thirion and X. Carbonelle attributed the motion to molecular fluctuations. They argued that the *distribution of molecular velocities will give rise to fluctuations in density and hence in pressure* in the microscopic scale which average out in the macroscopic scale. The idea of fluctuations was once again introduced by Leon Guoy in 1888 and found a supporter in none other than Henri Poincare. Scientists studied the effects of *solvent viscosity* and *ambient temperature* to find that for *finer particles* and *less viscous* and *hotter solvents* the motion was very pronounced.

Another very interesting, important and far-reaching implication came from the velocity measurements of the Brownian particles. The velocity did not seem to have a proper limit for small time intervals. The clue to the problem was in the fact that the Brownian trajectory was continuous but non smooth at every length scale. Smoluchowski actually understood that and held that the force acting on the Brownian particles was also non smooth. Assuming this, he arrived at results that could explain the behaviour of diffusing particles e.g sugar cubes diffusing in tea, perfume permeating from one corner of the room to another or the diffusion of a drop of ink in water. The crux of the problem lays in the understanding that the time scale of observation is much larger compared to the time scale at which the actual collisions are taking place. *So that the observed displacement is in reality an average over a multitude of zigzag displacements.* Thus, making the curve non-differentiable and rendering the concept of velocity or average velocity meaningless. This observation and understanding led to the modeling of Brownian motion by a random process and eventually gave birth to the theory of *probability calculus* and *the theory of stochastic process*.

Mathematicians knew about curves statistically self-similar at every scale, called *fractals* which are full of kinks and are differentiable nowhere. And Weierstrass had discovered such pathological functions like

$$g(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

where a is odd, $b \in (0,1)$, and $ab > 1 + 3\pi/2$ is nowhere differentiable. With Norbert Wiener showing in 1923 the mathematical existence of Brownian motion, existence of a random (*stochastic*) process with the given properties was truly established.

Together Einstein and Smoluchowski showed that viscosity and other forms of dissipation, are on a molecular level, caused by thermal motion of particles i.e. they found a relation between *viscous or any kind of dissipative force* and the *random or fluctuating part of the force* - the so called *fluctuation-dissipation* formula. Later Smoluchowski established that for a large but finite system in thermal equilibrium variables must vary in time in a manner akin to the Gaussian White Noise - the cause of Brownian motion; that is, he proved that **Brownian motion is ubiquitous in all macroscopic physical systems in equilibrium!!**

In fact, with the French physicist J. Perrin receiving the Nobel Prize in 1926, marking the beginning of the centenary of Brownian motion, for showing that the colliding particles obey gas laws and calculating the Avogadro's number, the phenomenon had come a full cycle and had a solid footing. Barring Kepler's laws, Newton's laws of dynamics and the laws of thermodynamics one rarely finds a phenomenon that is as relevant even today as Brownian motion, in spite of being almost two hundred years old.

2 The Mathematical Structure

Different theoretical approaches were developed by Einstein-Smoluchowski, Langevin, Lorentz and Fokker-Planck. We shall leave aside the Fokker-Planck approach for the sake of brevity. We restrict ourselves to the linearized picture for the sake of simplicity and also for the reason that motions along different directions are independent of each other. We notice that the *only* observable quantity is the displacement of the Brownian particle $s(t) = x(t) - x(0)$. We show that the mean squared displacement is proportional to time i.e. $\langle s(t)^2 \rangle = 2Dt$. And it turns out that the distribution of *s* is Gaussian or

$$W(s.t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{s^2}{4Dt}\right) \quad \text{... (2.0.1)}$$

where *D* is the diffusion constant. In the same way rotations about an axis can be considered with an analogous result which is left as an exercise to the readers.

2.1 The Einstein Approach: Comparison With Diffusion

2.1.1 Phenomenology

Suppose that

- the suspended particles, visible under microscope, are ***irregularly dispersed*** in a liquid.
- Movement of one particle is independent of the motion of others.

- Movement of one particle during ***two distinct and non-overlapping*** intervals of time are considered independent as long as the time interval is not too small.

We, therefore, introduce a characteristic time scale τ in the problem, by hand, which is microscopically large but macroscopically small, such that the motions during ***non-overlapping*** intervals of length τ can be considered statistically independent. Assuming (well substantiated by experiments) suspended particles exert a pressure

$$p = nk_BT \qquad \text{... (2.1.1)}$$

where n is the number density of the particles, T is temperature in degree Kelvin and k_B is the Boltzmann's constant, we can write phenomenologically

$$\vec{j} = -D\vec{\nabla}n \qquad \text{... (2.1.2)}$$

where $\vec{j}$ is the current density. Along x direction, the equation looks like

$$j = -D\frac{\partial n}{\partial x} \qquad \text{... (2.1.3)}$$

If, now an external force f_e and a velocity dependent resistive force $-\zeta v$ act on the particle, then in the stationary case the two forces balance each other giving rise to a terminal velocity

$$v = \frac{f_e}{\zeta} \qquad \text{... (2.1.4)}$$

leading to

$$j = nv = n\frac{f_e}{\zeta} \qquad \text{... (2.1.5)}$$

In a stationary state, where the pressure gradient causes current flow, the force density turns out to be

$$nf_e = -\frac{\partial p}{\partial x} = -k_BT\frac{\partial n}{\partial x} \qquad \text{... (2.1.6)}$$

leading to

$$j = -\frac{k_B T}{\zeta}\frac{\partial n}{\partial x} \qquad \ldots(2.1.7)$$

with the identification

$$D = \frac{k_B T}{\zeta} \qquad \ldots(2.1.8)$$

2.1.2 The Model

If in a time interval τ the x coordinate of a particle changes by s and the number density of particles being in the interval s to $s + ds$ is

$$dn = n\phi(s)ds \qquad \ldots(2.1.9)$$

where $\phi(s)$ is the probability that the x coordinate of the particle suffers a change s with the properties

$$\int_{-\infty}^{\infty} \phi(s)ds = 1 \quad \text{(normalization)} \qquad \ldots(2.1.10)$$

$$\phi(s) = \phi(-s) \quad \text{(isotropy)} \qquad \ldots(2.1.11)$$

Therefore, we can write,

$$n(x, t + \tau) = \int_{-\infty}^{\infty} n(x - s, t)\phi(s)ds \qquad \ldots(2.1.12)$$

It is extremely interesting to note that the above equation is, in essence, the Chapman-Kolmogorov equation! This is a very general equation that was formally established by Chapman and Kolmogorov much later and is considered a cornerstone of statistical processes. Einstein used it in his work even before its proponents established it!! Now assuming $n(x,t)$: to be a smooth function of its arguments (otherwise density could not be defied) and Taylor expanding both sides, we obtain

$$n(x,t) + \tau \frac{\partial n}{\partial t} + \cdots$$
$$= n(x,t) \int_{-\infty}^{\infty} \phi(s)ds - \frac{\partial n}{\partial x} \int_{-\infty}^{\infty} s\phi(s)ds$$
$$+ \frac{1}{2} \frac{\partial^2 n}{\partial x^2} \int_{-\infty}^{\infty} s^2 \phi(s)ds + \cdots \qquad \ldots (2.1.13)$$

The first terms on both sides cancel out, the second term on the right vanishes due to eqn. (2.1.11), ultimately, giving rise to upto the quadratic level,

$$\frac{\partial n}{\partial t} = \frac{\langle s^2 \rangle}{2\tau} \frac{\partial^2 n}{\partial x^2} \qquad \ldots (2.1.14)$$

Making use of the continuity equation,

$$\frac{\partial n}{\partial t} + \vec{\nabla}.\vec{j} = 0$$ or in one dimension

$$\frac{\partial n}{\partial t} + \frac{\partial j}{\partial x} = 0$$ leading to the equation

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} \qquad \ldots (2.1.15)$$

This is the famous partial differential equation, first order in time and second order in space, called the *Diffusion* equation. It is interesting to note the Schrödinger equation in non-relativistic quantum mechanics, that governs the time evolution of the quantum state, is also a diffusion equation, albeit in imaginary time. While in Brownian motion we deal with thermal fluctuations in quantum mechanics we encounter quantum fluctuations. Comparison between equation (2.1.14) and (2.1.15) yields

$$D = \frac{\langle s^2 \rangle}{2\tau} \qquad \ldots (2.1.16)$$

and

$$\langle s^2 \rangle = \frac{2k_BT}{\zeta}\tau \qquad \text{... (2.1.17)}$$

We understand that eqns.(2.1.12) and (2.1.15) represent the same physics, one in integral form and the other in the differential form. It must be kept in mind though, that in obtaining eqn.(2.1.15) we had to truncate eqn.(2.1.12) after Taylor expansion.

2.1.3 Calculation of The Distribution Function

Next, we take up the initial value problem of how $n(x, t)$ spreads in space and time as promised. We take the initial condition that at $t = 0$ all particles were placed at the origin rendering $n(x, t)$ to be infinite and zero otherwise. If we denote by N the total number of particles then

at $t = 0$ $\quad n(x,0) = N\,\delta(x)$ ... (2.1.18)

using the Fourier representation we can write

$$n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{n}(k,t)e^{ikx}dk \qquad \text{... (2.1.19)}$$

Using eqn.(2.1.19) in eqn.(2.1.15) we obtain

$$\frac{\partial \tilde{n}(k,t)}{\partial t} = -k^2\, D\tilde{n}(k,t) \qquad \text{... (2.1.20)}$$

yielding a solution

$$\tilde{n}(k,t) = \tilde{n}(k,o)e^{-Dk^2t} \qquad \text{... (2.1.21)}$$

Identifying $\tilde{n}\,(k, 0) = N$ we write

$$\tilde{n}(k,t) = Ne^{-Dk^2t} \qquad \text{... (2.1.22)}$$

and then using eqn.(2.1.22) in eqn.(2.1.19) we obtain

$$n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ne^{-Dk^2t}e^{ikx}dk = \frac{N}{2\pi}e^{-x^2/2Dt}\int_{-\infty}^{\infty} e^{-Dt\left(k-\frac{ix}{2Dt}\right)^2}dk$$

$$= \frac{N}{2\pi} e^{-\frac{x^2}{4Dt}} \sqrt{\frac{\pi}{Dt}} = \frac{N}{\sqrt{4\pi Dt}} e^{-x^2/4Dt} \qquad \text{... (2.1.23)}$$

We are now in a position to calculate the mean squared displacement with the above distribution formula

$$\langle x^2 \rangle = \frac{1}{N} \int_{-\infty}^{\infty} n(x,t) x^2 dx = \frac{1}{\sqrt{4\pi DT}} \int_{-\infty}^{\infty} x^2 e^{-x^2/4Dt} dx$$

$$= \frac{4Dt}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 \, e^{-x^2} dx = \frac{4Dt}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^{1/2} \; e^{-x} dx$$

$$= \frac{4Dt}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = 2DT \qquad \text{... (2.1.24)}$$

Defining the mean displacement as $l_x \equiv \sqrt{\langle x^2 \rangle} = \sqrt{2Dt}$ we see that it is proportional to the square root of time which is the hallmark of diffusion.

2.1.4 Some Idea About The Length And Time Scales Involved In The Process

For spherical particles having mass *m*, radius *r*, suspended in a fluid of viscosity ´ and moving under an external force f_e the resistive force is known to be $6\pi\eta r v_c$ (Stokes' Law) where v_c is the terminal velocity of the particle. This yields $v_c = = \frac{fe}{6\pi\eta r}$ and $\zeta = 6\pi\eta r.$ Starting with

$$D = \frac{k_B T}{\zeta} = \frac{RT}{A_0} \frac{1}{6\pi\eta r} \equiv \frac{RT}{A_0} B \qquad \text{... (2.1.25)}$$

where A_0 is the Avogadro's number and *B* is the mobility and *R* is the universal gas constant we see that this relation can be used to determine either *D* or A_0 and also that *D* depends only on η when temperature and radius of the suspended particles are known. Recalling the definition of

l_x, the mean displacement, we see that rewriting things properly we can express

$$l_x = \sqrt{\frac{RT}{A_0}\frac{1}{3\pi\eta r}} \qquad \dots (2.1.26)$$

or

$$A_0 = \frac{1}{l_x^2}\frac{RT}{3\pi\eta r} \qquad \dots (2.1.27)$$

Thus, if A_0 is known one can determine l_x and vice-versa. Using the known values of $A_0 = 6.023 \times 10^{23}, \eta = 1.35 \times 10^{-2} gm.cm^{-1}.sec^{-1}$ for water at $17°C$, radius $r = 10^{-4}$cm and $R = 8.31\times 10^7\ erg.mole^{-1}.deg.K^{-1}$ one finds

$l_x = 8 \times 10^{-5} cm$

2.1.5 Justifying The Existence of Random Force From Theoretical Considerations

From the kinetic theory results one can write for the suspended particles having a mass 'm'

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}k_B T \qquad \dots (2.1.28)$$

independent of their size and environment. For particles having $m = 2.5 \times 10^{-15}$ gm in colloidal platinum solution one calculates for $T = 292K$

$$\sqrt{\langle v^2\rangle} = \sqrt{\frac{3RT}{mA_0}} = 8.6 \text{ cm/ sec} \qquad \dots (2.1.29)$$

which gives us a rough idea about the order of magnitude of the velocity for thermal motion. In a solution of solvent viscosity η these particles will experience a resistive force according to the Stokes' Law and we have

$$m = \frac{dv}{dt} = -6\pi\eta r v \qquad \text{... (2.1.30)}$$

with a solution

$$v \;=\; v_0 \;\exp\left(-\frac{6\pi\eta r}{m}t\right) \qquad \text{... (2.1.31)}$$

If we denote by $t_{0.1}$ the time for v to become $0.1v_0$ then

$$t_{0.1} \;= \frac{m \ln 10}{6\pi\eta r} \qquad \text{... (2.1.32)}$$

For platinum particles in water we obtain $t_{0.1} \;=\; 3.3 \times \; 10^{-7}$ sec which implies that under the action of viscous forces only, the particles would almost completely lose their initial velocities within a time span of $t_{0.1}$. Therefore, the particles must get impulses from the water molecules at random during that time interval to sustain their average velocity. So, only by assuming the existence of a random force can one reconcile kinetic theory results with hydrodynamic predictions.

2.2 The Langevin Approach

We pick up the thread from where we left in the last section by writing down the equation of motion of the suspended particle in the form

$$m\frac{dv}{dt} = -\zeta v + F(t) \qquad \text{... (2.2.1)}$$

where *F*(*t*) is a random force or additive noise. Rewriting the equation in terms of position we obtain

$$m\ddot{x} = -m\Gamma\dot{x} + F(t)$$

$$\ddot{x} = -\Gamma\dot{x} + f(t) \quad \text{or}$$

$$\dot{v} = -\Gamma v + f(t) \qquad \text{... (2.2.2)}$$

where $\Gamma = \zeta/m$ and $f(t) = F(t)/m$. We assume

- $\langle F(t)\rangle \neq 0$ and $F(t)$ is independent of x and v.
- $F(t)$ varies extremely rapidly compared to variations of v.
- $\langle F(t)^2\rangle \neq 0$ and $\langle F(t)F(t')\rangle = 2D\delta(t-t')$where D is some constant.

It is interesting to note that the average or the correlation function $\langle F(t)F(t')\rangle$ is not only peaked about $t = t'$ but also a function of the time interval $|t-t'|$ only i.e. ***stationary in time*,** meaning that while performing a Brownian motion experiment any instant of time can be chosen to be the origin of time.

Taking an average of the last step of eqn.(2.2.2) we obtain

$$\frac{d}{dt}\langle v\rangle = -\Gamma\langle v\rangle \qquad \text{leading to a solution}$$

$$\langle v(t)\rangle = \langle v(0)\rangle e^{-\Gamma t} \qquad \ldots(2.2.3)$$

while using $\dot{x} = v$ we rewrite the equation as

$$\ddot{x} = -\Gamma\dot{x} + f(t) \qquad \ldots(2.2.4)$$

Multiplying the above equation by x, rearranging terms and taking an average we obtain

$$\frac{d}{dt}\langle x\dot{x}\rangle - \langle \dot{x}^2\rangle = -\frac{\Gamma}{2}\frac{d}{dt}\langle x^2\rangle + \langle xf(t)\rangle \qquad \ldots(2.2.5)$$

and then using the equipartition theorem in one dimension

$$\frac{1}{2}m\langle \dot{x}^2\rangle = \frac{1}{2}k_BT \qquad \ldots(2.2.6)$$

alongwith

$$\langle x\, f(t)\rangle = 0 \qquad \ldots(2.2.7)$$

yield

$$\left(\frac{d^2}{dt^2} + \Gamma\frac{d}{dt}\right)\langle x^2\rangle = 2\frac{k_BT}{m}. \qquad \ldots(2.2.8)$$

Equation (2.2.7) needs a bit of justification. The timescale at which the random force changes is extremely fast compared to the timescale over which an observable displacement is produced. So if we average out the fast degrees of freedom (an idea which is essentially used in the *Dynamic Renormalization Group Calculations*) in that process x can be thought of as essentially constant and the average is performed over only $f(t)$ yielding zero. It is in this light that the eqn.(2.2.7) has to be understood.

Solving the differential eqn.(2.2.8) we obtain,

$$\langle x^2(t)\rangle = \left(\frac{2k_BT}{m\Gamma}\right)t - \frac{2k_BT}{m\Gamma^2}(1-e^{-\Gamma t}). \qquad \text{... (2.2.9)}$$

We immediately notice that for large time scales (the ***diffusive regime***)

$$\langle x^2(t)\rangle \approx \left(\frac{2k_BT}{m\Gamma}\right)t \qquad \text{... (2.2.10)}$$

ignoring the constant term while for small enough times (the ***ballistic regime***)

$$\langle x^2(t)\rangle \approx \left(\frac{k_BT}{m}\right)t^2. \qquad \text{... (2.2.11)}$$

The timescale for which the behaviour of the system crosses over from the ballistic to the diffusive regime is extremely interesting. It is also important to note that while in the diffusive regime the dissipative force (represented by Γ) plays an important part in the ballistic regime the dynamics is governed entirely by the inertial motion. Comparing with eqn.(2.1.25) one immediately identifies $D = \frac{k_BT}{m\Gamma}$ as predicted by the Einstein's theory.

2.2.1 The Relation Between Random and Fluctuating Force: The Fluctuation-Dissipation Theorem

In this section we would like to establish a relation between the dissipative force and the randomly fluctuating force. To that end we start by multiplying the last step of equation (2.2.2) by v to obtain

$$v\dot{v} = -\Gamma v^2 + vf(t)$$

or

$$\frac{d}{dt}v^2 = -2\Gamma v^2 + 2v(t)f(t). \qquad \text{... (2.2.12)}$$

Now taking an ensemble average of eqn.(2.2.12) we obtain

$$\frac{d}{dt}\langle v^2\rangle = -2\Gamma\langle v^2\rangle + 2\langle v(t)f(t)\rangle \qquad \text{... (2.2.13)}$$

The last term in the above equation is entirely nontrivial. To deal with this term we start with the identity

$$\int_{t_1}^{t_2}\dot{v}(t')dt' = v(t_2) - v(t_1) \qquad \text{... (2.2.14)}$$

Setting $t_2 = t$ and $t_1 = t - \Delta t$ we get

$$v(t) = v(t-\Delta t) + \int_{t-\Delta t}^{t}\dot{v}(t')dt' \qquad \text{... (2.2.15)}$$

Substituting for $\dot{v}$ from equation (2.2.2) we get

$$v(t) = v(t-\Delta t) + \int_{t-\Delta t}^{t}[-\Gamma v(t') + f(t')]dt' \qquad \text{... (2.2.16)}$$

This implies

$$\langle v(t)f(t)\rangle = \langle v(t-\Delta t)f(t)\rangle + \int_{t-\Delta t}^{t}\langle[-\Gamma v(t') + f(t')]f(t)\rangle\, dt'$$

$$= -\Gamma\int_{t-\Delta t}^{t}\langle v(t')f(t)\rangle\, dt' + \int_{t-\Delta t}^{t}\langle f(t')f(t)\rangle\, dt' \qquad \text{... (2.2.17)}$$

It is important to understand that

$\langle v(t-\Delta t)\, f(t)\rangle = 0$

and also

$$\int_{t-\Delta t}^{t} \langle v(t')f(t)\rangle\, dt' = 0$$

because the random force at a time instant *t* can in no way be depend on the velocity at a previous instant. Therefore, we have from eqn.(2.2.17)

$$\langle v(t)\, f(t)\rangle = \int_{t-\Delta t}^{t} \langle f(t')\, f(t)\rangle\, dt' \qquad \text{... (2.2.18)}$$

We now use the stationary property of the correlation function which implies

$\langle f(t'+s)\, f(t+s)\rangle = \langle f(t')f(t)\rangle$

to obtain

$$\langle v(t')\, f(t)\rangle = \int_{t-\Delta t}^{t} \langle v(t')\, f(t)\rangle dt' = \frac{1}{2}\int_{t-\Delta t}^{t+\Delta t} \langle f(t')f(t)\rangle\, dt'$$

(and using $t' \equiv t+s$)

$$= \frac{1}{2}\int_{t-\Delta t}^{t+\Delta t} \langle f(t)f(t+s)\rangle ds = \frac{1}{2}\int_{-\Delta t}^{\Delta t} \langle f(0)f(s)\rangle ds$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \langle f(0)f(s)\rangle ds \qquad \text{... (2.2.19)}$$

where we have made use of the stationary property of the correlation in the 3rd step of the above equation. As s denotes a very small time scale and Δt is actually very large compared to it we have used ∞ in the limits of the integration in the last step of eqn.(2.2.19). Therefore, we return to eqn. (2.2.13)

$$\frac{d}{dt}\langle v^2\rangle = -2\Gamma\langle v^2\rangle + 2\langle v(t)\, f(t)\rangle$$

$$= -2\Gamma\langle v^2\rangle + \int_{-\infty}^{\infty} \langle f(0)\, f(s)\rangle ds \qquad \text{... (2.2.20)}$$

In thermal equilibrium,

$$\langle v^2 \rangle = \frac{k_B T}{m}$$

and hence

$$\frac{d}{dt}\langle v^2 \rangle = 0$$

leading to

$$\Gamma = \frac{m}{2k_B T}\int_{-\infty}^{\infty} \langle f(0)\, f(s)\rangle ds$$

$$= \frac{1}{2mk_B T}\int_{-\infty}^{\infty} \langle F(0)\, F(s)\rangle\, ds \qquad \text{... (2.2.21)}$$

The above equation is extremely important as it shows a critical balance in the system in equilibrium. On one hand energy is fed into the system through fluctuations and on the other it gets dissipated through the dissipative mechanisms operative in the system. Only in equilibrium they strike a balance. This is the famous ***Fluctuation-Dissipation Relation***. On the left hand side of eqn.(2.2.21) we have $\Gamma = \frac{\zeta}{m}$, the scaled co-efficient of the velocity dependent dissipative force and on the other we have a two point correlation function of the fluctuating force. The special case where the fluctuating forces are δ −ccorrelated we have a very simplified picture where

$$\Gamma = \frac{1}{2mk_B T}\int_{-\infty}^{\infty} 2D\delta(s)ds = \frac{D}{mk_B T} \qquad \text{... (2.2.22)}$$

Here we have made use of the 3rd property of the fluctuating force $F(t)$ as stated in the beginning of the section (2.2). Recalling that $\zeta = 6\pi\eta r$ for the Stokes' Law we readily see the connection

$$D = 6\pi\eta r\, k_B \mathrm{T} \qquad \text{...(2.2.23)}$$

which is a simplified form of Fluctuation-Dissipation Relation in a special situation.

2.3. The Lorentz Approach

This is an innovative iterative procedure devised by H.A. Lorentz to depict the same results obtained by Einstein's theory but starting with the Langevin equation in velocity space i.e. the last step of eqn.(2.2.2). Assuming the initial condition that at $t = 0$ the velocity is v_0 and that at t is v_t we write

$$\dot{v} = -\Gamma v + f(t)$$

and integrating w.r.t time from 0 to t obtain

$$v_t - v_0 = -\Gamma\, v_0 t + h(t) + O(t^2) \qquad \text{... (2.3.1)}$$

where we have integrated over a sufficiently small time interval so that the terms quadratic in time can be neglected and

$$h(t) = \int_0^t f(t')dt' \qquad \text{... (2.3.2)}$$

Rearranging and rewriting eqn.(2.3.1) we get

$$v_t = v_0(1 - \Gamma t) + h(t) \qquad \text{... (2.3.3)}$$

Squaring eqn.(2.3.3) and keeping terms linear in time yields

$$v_t^2 = v_0^2(1 - 2\Gamma\, t) + 2v_0(1 - \Gamma\, t)\, h(t) + h(t)^2 + O(t^2) \qquad \text{... (2.3.4)}$$

Taking the ensemble average of eqn.(2.3.4) with the knowledge that

$$\langle v_0^2 \rangle = \langle v_t^2 \rangle \qquad \text{... (2.3.5)}$$

because of thermal equilibrium and

$$\langle v_0 h(t) \rangle = 0 \qquad \text{... (2.3.6)}$$

we obtain

$$\langle h(t)^2 \rangle = \left(\frac{2\Gamma k_B T}{m}\right) t. \qquad \text{... (2.3.7)}$$

Now consider a total time interval $\tau = nt$ split into n equal intervals of duration t such that in the second step the variables are v_1 and h_1, in the third they are v_2 and h_2, in the fourth v_3 and h_3 and so on. Defining

$\beta = (1 - \Gamma\, t)$

we rewrite eqn.(2.3.3) and use it to write a system of equations as

$v_1 = \beta v_0 + h_1(t)$

$v_2 = \beta v_1 + h_2(t) = \beta^2 v_0 + \beta h_1(t) + h_2(t)$

$v_3 = \beta v_2 + h_3(t) = \beta^3 v_0 + \beta^2 h_1(t) + \beta h_2(t) + h_3(t)$

.

.

$v_n = \beta v_{n-1} + h_n(t) = \beta^n v_0 + \beta^{n-1} h_1 + \beta^{n-2} h_2 + \cdots + h_n(t)$... (2.3.8)

Denoting $v_0 = h_0$ for the sake of notational convenience we can in general write

$$v_n = \sum_{k=0}^{n} \beta^{n-k} h_k \qquad \text{... (2.3.9)}$$

The displacement Δx produced in time τ therefore, can be expressed as

$$\Delta x = t(v_0 + v_1 + v_2 + \cdots + v_{n-1})$$

$$= t[v_0 + \big(\beta v_0 + h_1(t)\big) + \big(\beta^2 v_0 + \beta h_1(t) + h_2(t)\big) + (\beta^3 v_0 + \beta^2 h_1(t) + \beta h_2(t) + h_3(t) + \cdots ...]$$

$$= t[v_0(1 + \beta + \beta^2 + \cdots + \beta^{n-1}) + h_1(1 + \beta + \beta^2 + \cdots + \beta^{n-2}) + h_2(1 + \beta + \beta^2 + \cdots + \beta^{n-3}) + \cdots + h_{n-1}]$$

$$= t\left[h_0 \frac{1-\beta^n}{1-\beta} + h_1 \frac{1-\beta^{n-1}}{1-\beta} + h_2 \frac{1-\beta^{n-2}}{1-\beta} + \cdots + h_{n-1}\right]$$

$$= \frac{t}{1-\beta}[h_0\,(1-\beta^n) + h_1\,(1-\beta^{n-1}) + h_2\,(1-\beta^{n-2}) + \cdots + h_{n-1}\,(1-\beta)]$$

$$= t \sum_{v=0}^{n-1} h_v \frac{(1-\beta^{n-v})}{(1-\beta)} \qquad \text{... (2.3.10)}$$

We now try to calculate $\langle(\Delta x)^2\rangle$ and accordingly write

$$\langle(\Delta x)^2\rangle = \frac{t^2}{(1-\beta)^2}\langle(h_0\,(1-\beta^n) + \sum_{v=1}^{n-1} h_v\,(1-\beta^{n-v})(h_0\,(1-\beta^n) + \sum_{\mu=1}^{n-1} h_\mu\,(1-\beta^{n-\mu}))\rangle$$

$$= \frac{t^2}{(1-\beta)^2}\left(\sum_{v,\mu=1}^{n-1}(1-\beta^{n-v})\,(1-\beta^{n-\mu})\langle h_v h_\mu\rangle + 2\sum_{v=1}^{n-1}\,(1-\beta^{n-v})\langle h_v h_0\rangle + (1-\beta^n)^2\langle h_0^2\rangle\right) \quad \text{... (2.3.11)}$$

We now make use of the relations

$$\langle h_v h_0\rangle = 0 \ \text{ for } \ v \neq 0 \quad \text{... (2.3.12)}$$

$$\langle h_\mu h_v\rangle = \delta_{\mu v}\langle h^2\rangle \ \text{ for } \ \mu, v \neq 0 \quad \text{... (2.3.13)}$$

$$\langle h^2\rangle = \left(\frac{2\Gamma k_B T}{m}\right) t \quad \text{... (2.3.14)}$$

$$\langle v_0^2\rangle = \langle h_0^2\rangle = \frac{k_B T}{m} \quad \text{... (2.3.15)}$$

Plugging in the above relations in eqn.(2.3.11) we get

$$\langle(\Delta x)^2\rangle = \frac{t^2}{(1-\beta)^2}\left(\sum_{v=1}^{n-1}(1-\beta^{n-v})^2\langle h^2\rangle + (1-\beta^n)^2\langle h_0^2\rangle\right)$$

$$= \frac{t^2}{(1-\beta)^2}\left(\sum_{v=1}^{n-1}(1-\beta^{v})^2\langle h^2\rangle + (1-\beta^n)^2\langle v_0^2\rangle\right)$$

$$\langle(\Delta x)^2\rangle = t^2\frac{(1-\beta^n)^2}{(1-\beta)^2}\langle v_0^2\rangle + \frac{t^2\langle h^2\rangle}{(1-\beta)^2}\sum_{v=1}^{n-1}(1-\beta^v)^2$$

$$= \frac{1}{\Gamma^2}\left[\langle v_0^2\rangle(1-\beta^n)^2 + \langle h^2\rangle \sum_{v=1}^{n-1}(1-\beta^v+\beta^{2v})\right]$$

$$= \frac{1}{\Gamma^2}\left[\langle v_0^2\rangle(1-\beta^n)^2 + \langle h^2\rangle(n-1-2\beta\frac{1-\beta^{n-1}}{1-\beta}\right.$$

$$\left.+\beta^2\frac{1-\beta^{n-2}}{1-\beta^2}\right] \qquad \text{... (2.3.16)}$$

Now in the limit of very large n only the term proportional to n matters and hence in that limit using eqn.(2.3.14) we get

$$\langle(\Delta x)^2\rangle \simeq \frac{n}{\Gamma^2}\langle h^2\rangle = \frac{n}{\Gamma^2}\left(\frac{2\Gamma k_B T}{m}\right)t$$

$$= \left(\frac{2\Gamma k_B T}{m\Gamma}\right)(nt)$$

$$= \left(\frac{2\Gamma k_B T}{\zeta}\right)\tau \qquad \text{... (2.3.17)}$$

the same result, obtained from both the Einstein and Langevin picture. It would be interesting if one could recover the result for the ballistic regime from this picture in the limit of very small n. One could even try to derive the Fluctuation-Dissipation relation from this picture.

2.4. Random Walk And The Brownian Motion

Consider an infinite one dimensional lattice with lattice spacing a with a drunkard situated at one of the lattice points trying to walk to his home. He is so drunk that he does not know his way and takes a step to the right or left with equal probability of $\frac{1}{2}$. His movement has two features though:

- he always takes a step of size unity
- the duration of a single step is Δt

We are interested to find the mean and variance of the displacement of the walker as a function of time. This is the standard random walk problem. If the drunkard takes steps in two different directions with two different probabilities then that would be the biased random walk problem. We are going to present a discrete time formalism of the problem and indicate how the passage to continuum time description may be effected.

2.4.1 Formulation of The Problem

Suppose after j-steps the walker lands up at the nth site. We define $P(n,j)$ as the probability for the walker to be at the nth site after the elapse of jth time step. Then, we understand that at the previous instant of time, i.e. at the $(j-1)$th instant, the walker was either at the $(n-1)$th or the $(n+1)$th site. therefore one writes

$$P(n,j) = \frac{1}{2}[P(n+1,j-1) + P(n-1,j-1)]$$

or

$$P(n,j+1) = \frac{1}{2}[P(n+1,j) + P(n-1,j)] \qquad \text{... (2.4.1)}$$

where we have obtained the second step by replacing j by $j+1$ in the first step. The symmetry of the unbiased random walk is reflected in the fact that the equations are invariant under the transformation $(n-1) \longleftrightarrow (n+1)$. The passage to the continuum description follows automatically by subtracting the first step from the second of eqn.(2.4.1) to obtain

$$P(n,j+1) - P(n,j) = \frac{1}{2}[P(n+1,j) + P(n-1,j) - 2P(n,j)]$$

$$\frac{P(n,j+1) - P(n,j)}{(j+1)\Delta t - j\Delta t} = \frac{1}{2}\frac{[P(n+1,j) + P(n-1,j) - 2P(n,j)]}{(j+1)\Delta t - j\Delta t}$$

$$\frac{P(n, j+1) - P(n, j)}{\Delta t}$$

$$= \frac{1}{2\Delta t}\,[P(n+1, j) + P(n-1, j) - 2P(n, j)] \quad \text{... (2.4.2)}$$

$$\frac{\partial P(n, t)}{\partial t} = \omega[P(n+1, t) + P(n-1, t) - 2P(n, t)] \quad \text{... (2.4.3)}$$

where to obtain eqn.(2.4.3) from eqn.(2.4.2) we have taken the limit $\Delta t \to 0$. The product $j\,\Delta t \to t$ when along with $j \to \infty$ we also impose $\Delta t \to 0$. $P(n, t)$ represents the probability that the drunken walker occupies the *n*th site when time t has elapsed. $\omega = (2\Delta t)^{-1}$ is the transition probability between a site and one of its nearest neighbours. One might wonder about what happens to ω in the limit $\Delta t \to 0$ but that is the price that we pay for using '*continuous time - discrete space*' description of the problem. The problem does not occur when both space and time are either continuous or discrete. One look at the r.h.s of the third step of eqn.(2.4.2) would tell anybody that the expression is nothing but the second partial derivative of $P(n, t)$ w.r.t space should we care to divide and multiply the expression by the lattice spacing and then take the limit of lattice spacing $\to 0$. The eqn. (2.4.3) is then, in disguise, the diffusion equation that we know so well by now. Mathematically speaking, the displacement of the drunken walker is a Markov process, where the probabilities are governed by the Master Equation (2.4.3).

2.4.2 Solution of The Problem

First, we tackle the problem using the idea of distribution. Assume the lattice to be lying along the x-direction. The total number of steps taken by the walker is j, out of which j_+ steps are taken in the forward direction while $j_{\_}$ are taken in the backward direction. This immediately yields

$j_+ + j_- = j$ and $j_+ - j_- = n$

or

$$j_+ = \frac{1}{2}(j+n) \text{ and } j_- = \frac{1}{2}(j-n) \quad \text{... (2.4.4)}$$

As the walker takes the forward and backward steps with equal probability we write

$$P(n,j) = \left(\frac{1}{2}\right)^j \frac{j!}{j_+!\,j_-!} = \left(\frac{1}{2}\right)^j \frac{j!}{\left(\frac{j+n}{2}\right)!\left(\frac{j-n}{2}\right)!} \quad \text{... (2.4.5)}$$

i.e. the probability distribution of the particles after j steps follows a binomial distribution. Taking the general case of the biased random walk as an example we can write

$P(n,j) = [p\,P(n-1,j-1) + q\,P(n+1,j-1)]$ and

$$p + q = 1 \quad \text{... (2.4.6)}$$

where p is the probability of the forward step while q is that of the backward step. In the same manner as before one can write down

$$P(n,j) = {}^{j}C_{j_+} p^{j_+} q^{j_-} = \frac{j!}{\left(\frac{j+n}{2}\right)!\left(\frac{j-n}{2}\right)!} p^{\frac{j+n}{2}} {}_{+} q^{\frac{j-n}{2}} \quad \text{... (2.4.7)}$$

In the limit of large j and large pj the distribution reduces to a Gaussian distribution after one makes use of the *Stirling's approximation*. Writing j_+ as x for convenience, we determine the mean and variance as

$$\langle x \rangle = \sum_{x=0}^{j} {}^{j}C_x x p^x q^{j-x}$$

$$= \sum_{x=0}^{j} \frac{j!}{x!\,(j-x)!}\; x\; p^x (1-p)^{j-x}$$

$$= \sum_{x=1}^{j} \frac{j!}{(x-1)!\,(j-x)!}\; p^x (1-p)^{j-x}$$

$$= j\,p \sum_{x=1}^{j-1} \frac{(j-1)!}{(x-1)!\,(j-x)!} p^{x-1} (1-p)^{(j-1)-(x-1)}$$

$$= j\,p \qquad \text{... (2.4.8)}$$

In a similar manner, the mean of the square can also be calculated using

$$\langle x^2 \rangle = \sum_{x=0}^{j} {}^{j}C_x x^2 p^x q^{j-x}$$

$$= \sum_{x=0}^{j} \frac{j!}{x!\,(j-x)!} x^2\, p^x (1-p)^{j-x}$$

$$= \sum_{x=0}^{j} \frac{j!}{(x-1)!\,(j-x)!} x\, p^x (1-p)^{j-x}$$

$$\langle x^2 \rangle = \sum_{x=1}^{j} \frac{j!}{(x-1)!\,(j-x)!} (x-1) p^x (1-p)^{j-x}$$

$$+ \sum_{x=1}^{j} \frac{j!}{(x-1)!\,(j-x)!} p^x (1-p)^{j-x}$$

$$= j(j-1)p^2 \sum_{x=2}^{j-2} \frac{(j-2)}{(x-2)!\,(j-x)!} p^{x-2} (1-p)^{j-x} + \langle x \rangle$$

$$= j(j-1)p^2 + jp. \qquad \text{... (2.4.9)}$$

Hence the variance is

$$\langle x^2 \rangle - \langle x \rangle^2 = j(j-1)p^2 + j\,p - j^2 p^2 = jp(1-p) \qquad \text{... (2.4.10)}$$

With these basic formalities completed we define the Fourier transform on the discrete lattice by

$$\tilde{P}(k,t) = \sum_x P(x,t)e^{ikx} \qquad \text{... (2.4.11)}$$

with j replaced by t, n by x and $P(n,j)$ by $P(x,t)$. Applying this to the master eqn.(2.4.3) we obtain

$$\frac{\partial}{\partial t}\tilde{P}(k,t) = \omega\left[e^{ik} + e^{-ik} - 2\right]\tilde{P}(k,t)$$

$$= -2\omega(1-\cos k)\,\bar{P}(k,t) \qquad \text{... (2.4.12)}$$

with a solution

$$\tilde{P}(k,t) = e^{-2\omega t(1-\cos k)}\tilde{P}(k,t) \qquad \text{... (2.4.13)}$$

The initial condition $\tilde{P}(k,0) = 1$ corresponds to $P(x,0) = \delta_{x,0}$ (ref. eqn. (2.1.18)). Now noting that

$$\langle x^2 \rangle = \sum_x x^2\, P(x,t)$$

$$= -\frac{\partial^2 \tilde{P}(k,t)}{\partial k^2}|_{k=0}$$

$$= 2\omega t \qquad \text{... (2.4.14)}$$

where we have used eqn.(2.4.11) in the second step and eqn.(2.4.13) in the third step of the above equation with the initial condition $\tilde{P}(k,0) = 1$. Once again, we obtain diffusion from the standard random walk. Using the inverse Fourier transform of eqn.(2.4.13) we obtain,

$$P(x,t) = \sum_x \tilde{P}(k,t)e^{-ikx} = \sum_k e^{-2\omega t(1-\cos k)}e^{-ikx}$$

$$= e^{-2\omega t}\sum_k e^{(2\omega t\cos k - ikx)} = e^{-2\omega t}I_x(2\omega t) \qquad \text{... (2.4.15)}$$

where $I_x(2\omega t)$ is the modified Bessel function of first kind and order x. Using the asymptotic behaviour of this function ($x \to \infty, t \to \infty$ with $\frac{x^2}{t}$ fixed) we obtain

$$P(x,t) \simeq \frac{1}{\sqrt{4\pi\,\omega t}} \exp\left(-\frac{x^2}{4\omega t}\right) \qquad \text{... (2.4.16)}$$

Comparing with eqn.(2.1.33) we immediately identify ω as D in the Einstein's formulation. To take care of the dimension of D and ω we recall that to begin with we took the lattice spacing to be unity for simplifying our calculations. If instead we worked with a as the lattice spacing then we would get ωa^2 in place of w and that would take care of the problem of dimensional incongruence.

2.4.3 Biased Random Walk And The Diffusion Equation with Drift Term

We consider the case of biased random walk defined by eqn.(2.4.6) on a lattice of lattice spacing a, time step Δt and write

$$P(n, j+1) = [p\,P(n-1, j) + q\,P(n+1, j)]$$

$$P(n, j) = [p\,P(n-1, j-1) + q\,P(n+1, j-1)] \text{ and}$$

$$p + q = 1. \qquad \text{... (2.4.17)}$$

Taking the difference of the first two steps and using the third we obtain

$$P(n, j+1) - P(n, j)$$

$$= p\,[P(n-1, j) - P(n, j)] + q\,[(P(n+1, j) - P(n, j)]$$

$$= \frac{1}{2}[P(n+1, j) + P(n-1, j) - 2\,P(n, j)]$$

$$-\frac{\delta}{2}[P(n+1, j) - P(n-1, j)] \qquad \text{... (2.4.18)}$$

where we have defined

$$p = \frac{1+\delta}{2}$$

$$q = \frac{1-\delta}{2} \qquad \text{... (2.4.19)}$$

It is clear that eqn.(2.4.18) is a difference equation where each side is written as finite difference. On the L.H.S the position index remains fixed while the temporal index changes by one unit while on the R.H.S the temporal index remains fixed the spatial index changes. Writing the equation in the following way

$$\frac{P(n,j\ +\ 1) - P(n,j)}{\Delta t}\Delta t$$
$$= \frac{a^2}{2}\frac{[P(n+1,j), +P(n-1,j) - 2P(n,j)]}{a^2}$$
$$- \frac{a\delta}{2}\frac{[P(n+1,j) + P(n-1,j)]}{a}$$

$$\frac{P(n,j+1) - P(n,j)}{\Delta t}$$
$$= \frac{a^2}{2\Delta t}\frac{[P(n+1,j) + P(n-1,j) - 2P(n,j)]}{a^2}$$
$$- \frac{a\delta}{2\Delta t}\frac{[P(n+1,j) - P(n-1,j)]}{a} \qquad \text{... (2.4.20)}$$

We now consider the limit $a \to 0, \Delta t\ \to 0,$ such that $\frac{a_2}{\Delta t} \to$ a constant in that limit. In this limit, the L.H.S is the partial derivative of P w.r.t time while the first term on the R.H.S is the second partial derivative of P w.r.t space. The remaining term gives a meaningful first partial derivative w.r.t space only if $\frac{a\delta}{\Delta t} \to$ another constant in the above limit implying that along with a and $\Delta t, \delta$ should also tend to 0 for the equation to have a proper continuum limit. Defining

$$\lim_{a,\Delta t\to 0}\left(\frac{a^2}{2\Delta t}\right) = D \text{ and}$$

$$\lim_{a,\Delta t,\delta\to 0}\left(\frac{a\delta}{2\Delta t}\right) = D_1 \qquad \ldots(2.4.21)$$

the continuum limit of eqn.(2.4.20) becomes

$$\frac{\partial}{\partial t}P(x,t) = D\frac{\partial^2}{\partial x^2}P(x,t) - D_1\frac{\partial}{\partial x}P(x,t) \qquad \ldots(2.4.22)$$

This is the old *diffusion* equation with an additional *drift* term. The drift arises due to the left-right asymmetry or the biasing; but it is interesting to notice that the coefficient D_1 is defined in the limit $\delta \to 0$ which implies that in that limit $p - q \approx 0$ or $p \approx q \approx 0.5$. This is a special form of the Fokker-Planck equation which we discuss in the next section.

3. Epilogue

The rich mathematical framework generated by this phenomenon is being extensively used in mathematics, biology, ecology, social sciences, stock market fluctuations, nonlinear dynamics, dynamic critical phenomena and in many other contemporary topics. The advent of quantum Brownian motion has expanded the horizon even further by giving rise to topics like quantum noise and its applications in the field of quantum optics and by establishing a link between dissipative systems and quantum field theory; ***quantum decoherence*** needs a special mention here as 'A Nobel Prize' in physics was awarded in this topic where noise plays a vital role. Thanks to Brownian motion we understand the role played by noise in our life - from Johnson's noise in the electrical circuits to the design of nanorobots to molecular motors (e.g. *kinesins* or proteins that move on intra molecular membranes)- it is everywhere. We now realize that *noise* is not always detrimental as it can act constructively not only by sustaining the signal but also by amplifying it (e.g. *stochastic resonance*); finally noise is a crucial element in all biochemical reactions without which life in its present form would not exist. So, for a complete understanding of life in particular as a scientific process and science in general, one must understand Brownian motion which, despite being almost two hundred years old, still remains relevant in the twenty first century.

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On Some Applications of Pseudo Cyclic Parallel RICCI Symmetric Manifolds

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[Abstract: In a resent paper the present author[1] introduced and studied a type of Riemannian manifold called pseudo cyclic parallel Ricci symmetric manifold. Some properties of this manifold have been obtained in Riemannian and semi Riemannian manifolds].

Key words : Torse-forming vector field, quasi Einstein manifold, perfect fluid space time of general relativity.

1. Introduction:

In a paper M. C. Chaki[2] introduced and studied a type of non-flat Riemannian manifold called pseudo Ricci symmetric manifold. According to him, a non-flat Riemannian manifold (M^n, g), $(n > 2)$, is called pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \quad \dots(1.1)$$

for every vector field X, Y, Z , where A is a non-zero 1-form defined by $g(X, U) = A(X), \nabla$ denotes the operator of covariant differentiation with respect to the metric tensor g. A is called associated 1-form and U is called associated vector field. An n-dimensional manifold of this kind was denoted by the symbol $(PRS)_n$.

In a recent paper the present author[1] introduced and studied a new type of non-flat Riemannian manifold called pseudo cyclic parallel Ricci symmetric manifold. A non-flat Riemannian manifold $(M^n, g), (n > 2)$, is called pseudo cyclic parallel Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = -2A(X)S(Y, Z) + A(Y)S(Z, X) + A(Z)S(X, Y) \quad \text{... (1.2)}$$

for every vector field X, Y, Z , where A is a non-zero 1-form defined by

$$g(X, U) = A(X) \quad \text{...(1.3)}$$

∇, A and U have the meaning already mentioned. An n-dimensional manifold of this kind is denoted by the symbol $(PCPRS)_n$. The main difference of these two manifolds is that in $(PRS)n$, the Ricci tensor Is not cyclic parallel but in $(PCPRS)_n$, the Ricci tensor is cyclic parallel. This can be easily verified from the definition of cyclic parallel Ricci tensor[3] as given below:

The Ricci tensor S of type (0, 2) is called cyclic parallel if it satisfies the condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0 \quad \text{... (1.4)}$$

So the name pseudo cyclic parallel Ricci symmetric manifold has been chosen.

The aim of these paper is to study some applications of $(PCPRS)_n$ in Riemannian and semi Riemannian manifolds.

2. Preliminaries

In this section we shall obtain some formulas which will be used in sequel.

Let (M^n, g) be a Riemannian manifold and { e_i}, i = 1, 2, ……, n be an orthonormal basis of the tangent space at each point and i is summed for $1 \leq i \leq n$. Let r be the scalar curvature[4] of the manifold and it is defined by

$S(e_i, e_i) = r.$ ... (2.1)

Let L be the symmetric endomorphism corresponding to the Ricci tensor S of type (0, 2) and is defined by

$g(LX, Y) = S(X, Y),$ ... (2.2)

Putting $Y = Z = e_i$ in (1.2) and i is summed for $1 \leq i \leq n,$ we get,

$dr(X) = -2A(X)r + 2S(X, U).$ ...(2.3)

From (1.2) we get

$(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) =$
$-3[A(X)S(Y, Z) - A(Z)S(X, Y)].$ ... (2.4)

Putting $Y = Z = e_i$ in (2.4) and i is summed for $1 \leq i \leq n,$ we get,$dr(X) = -6[A(X)r - S(X, U)].$ ... (2.5)

From (2.3) and (2.5) we have

$S(X, U) = A(X)r.$ ... (2.6)

From (2.3) and (2.6) we get$dr(X) = 0.$ ... (2.7)

3. Associated vector field U of $(PCPRS)_n$ as a torse forming vector field :

In this section we find the consequences if the associated vector field U of $(PCPRS)_n$ is a torse forming vector field.

Let U be a torse forming vector field[5], then

$\nabla_X U = aX + \omega(X)U,$ ... (3.1)

where a is a non-zero scalar and ω is a non-zero 1-form.

We consider associated 1-form of $(PCPRS)_n$ is the associated 1-form of the torse forming vector field ρ,$A(X) = \omega(X)$. ...(3.2)

From (3.1) and (3.2) we get

$\nabla_X U = aX + A(X)U.$ ...(3.3)

From (2.5) and (2.6) we get

$rdA(X,Y) = (\nabla_X S)(Y,U) - (\nabla_Y S)(X,U) + S(Y,\nabla_X U) - S(X,\nabla_Y U).$ ... (3.4)

Putting $Y = U$ in (2.4) and using (2.5) we get

$(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U) = 0.$ ... (3.5)

From (3.3), (3.4) and (3.5) we get

$rdA(X,Y) = 0.$ ... (3.6)

From (3.6) we see that either the scalar curvature is zero or for non-zero scalar curvature the associated 1-form A is closed .

Hence we can state the following theorem:

Theorem 2.1: In a $(PCPRS)_n$ with associated vector field U defined by (3.3), either the scalar curvature is zero or for non-zero scalar curvature the associated 1-form A is closed .

To a vector field we attach the real C^∞ function f defined by

$f = \frac{1}{2}g(U,U).$ ... (3.7)

This function f is called the energy function of the vector field U.

$$df(Y) = Yf = Y\frac{1}{2}g(U,U) = g(\nabla_Y U,U)$$

$$= g(aY + A(Y)U,U) \text{ [by (3.3)]}$$

$= ag(Y, U) + A(Y)2f$

$= (a + 2f)A(Y).$ ... (3.8)

Putting $Y = U$ in (3.8) and using (3.7) we get

$df(U) = 2(a + 2f)f.$ ... (3.9)

From (3.9) we conclude that the critical points[6] of the energy f of the vector field U are either zeros of f or zeros of $a + 2f$. This leads to the following theorem:

Theorem 2.2: In a $(PCPRS)_n$ with associated vector field U defined by (3.3), the critical points of the energy f of the vector field U are either zeros of f or zeros of $a + 2f$.

If in particular f is constant, it follows from (3.7) and (3.8)

$a = -A(U)\,.$ ... (3.10)

From (3.3) and (3.10) we get

$\nabla_X U = - A(U)X + A(X)U.$ ... (3.11)

Putting X = U in (3.11) we get

$\nabla_U U = 0.$ ... (3.12)

Hence we can state the following theorem:

Theorem 2.3: If in a $(PCPRS)_n$ with associated vector field U defined by (3.3), the energy f of the vector field U is constant, then the integral curves of U are geodesics.

4. Semi Riemannian $(PCPRS)_4$

Let a semi Riemannian $(PCPRS)_4$ be a general relativistic space time (M^4, g), where g is a Lorentz metric with signature $(+,+,+.-)$. We consider general relativistic perfect fluid space time (M^4, g) with unit time like velocity vector field as the associated vector field U of $(PCPRS)_4$ i.e.,$g(U,U) = -1$. ...(4.1)

The sources of any gravitational field (matter and energy) are represented in relativity by a type of (0,2) symmetric tensor T called the energy momentum tensor[7,8,9, 10] . T is given by

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \quad \text{... (4.2)}$$

where σ and p are the energy density and the isotropic pressure of the fluid respectively, while A is defined by

$$g(X,U) = A(X). \quad \text{... (4.3)}$$

For a perfect fluid space time, Einstein equation without cosmological constant is as follows:

$$S(X,Y) - \frac{1}{2}rg(X,Y) = KT(X,Y), \quad \text{... (4.4)}$$

where K is a gravitational constant.

From (4.2) and (4.4) we have

$$S(X,Y) - \frac{1}{2}rg(X,Y) = K[(\sigma + p)A(X)A(Y) + pg(X,Y)]. \quad \text{... (4.5)}$$

Let $\{e_i\}, i = 1, 2, 3, 4$ be an orthonormal basis of the frame field at a point of the space time and contracting (4.5) we get

$$r = K(\sigma - 3p). \quad \text{...(4.6)}$$

If $r \neq 0$ we have from theorem 2.1$dA(X,Y) = 0$ ...(4.6 a)

which gives $g(\nabla_X U,Y) - g(X,\nabla_Y U) = 0$. ... (4.7)

From (4.1) we have $g(\nabla_X U,U) = 0$. ... (4.8)

putting $Y = U$ in (4.7) and using (4.8) we get

$$g(X,\nabla_U U) = 0, for\ all\ X. \quad \text{...(4.9)}$$

Hence $\nabla_U U = 0$. Since U is the velocity vector field of the $(PCPRS)_4$ space- time, it follows that for a perfect fluid $(PCPRS)_4$, the fluid has zero acceleration and the integral curves of the velocity vector field are geodesics. This leads to the following theorem:

Theorem 3.1: In a perfect fluid $(PCPRS)_4$ space-time with time like velocity vector field defined by (3.3) having Einstein equation without cosmological constant, the fluid has zero acceleration and the integral curves of the velocity vector field are geodesics.

Putting $Y = U$ in (4.5) and using (4.1) and (4.6) we get

$S(X,U) = -\frac{K}{2}(\sigma + 3p)A(X).$ ...(4.10)

From (2.6) and (4.6) we get $S(X,U) = K(\sigma - 3p)A(X).$ ... (4.11)

From (4.10) and (4.11) we get $\sigma = p.$ ... (4.12)

From (4.5), (4.6) and (4.12) we get

$S(X,Y) = 2KpA(X)A(Y).$ ...(4.13)

We know that , if the energy momentum tensor T of the space- time obeys the time like convergence condition[9] , then the Ricci tensor of the space- time satisfies the condition

$S(X,X) > 0$ for every time like vector field X. ...(4.14)

Putting $X = Y = U$ in (3.13) and using (4.1), (4.12) and (4.14) we get

$$\sigma = p = \frac{S(U,U)}{2K} > 0 \qquad \text{... (4.15)}$$

Hence in this space- time pure matter exists. This leads to the following theorem:

Theorem 3.2 : In a perfect fluid $(PCPRS)_4$ space-time having Einstein equation without cosmological constant, if the energy momentum tensor T of the space- time obeys the time like convergence condition , then the space-time under consideration contains pure matter.

In the year 2000, Chaki and Maity[11] introduced the notion of quasi Einstein manifold whose Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$S(X,Y) = \alpha\, g(X,Y) + \beta\, A(X)A(Y),$ ...(4.16)

where α, β are scalars of which $\beta \neq 0$ and A is a non-zero 1-form defined by $g(X,U) = A(X), U$ being a unit vector field on the manifold. Comparing (4.13) and (4.16) we see that this space-time is a special type of quasi Einstein manifold with $\alpha = 0$ and $\beta = 2Kp \neq 0.$ This leads to the following theorem:

Theorem 3.3: A perfect fluid $(PCPRS)_4$ space-time having Einstein equation without cosmological constant is a special type of quasi Einstein manifold.

Putting $Y = U$ in (3.13) and using (4.1) we get

$S(X,U) = -2Kp\, g(X,U).$ ... (4.17)

It follows from (4.17) that $-2Kp$ is an eigenvalue of the Ricci tensor S and U is an eigenvector corresponding to this eigenvalue.

Let V be another eigenvector of S different from U. Then V is orthogonal to U. Hence

$g(U,V) = 0, or,\ A(V) = 0.$ ... (4.18)

Putting $Y = V$ in (4.13) and using (4.18) we get

$S(X,V) = 0.$ ... (4.19)

From (4.19), it follows that 0 is another eigenvalue of S corresponding to the eigenvector V. Let the multiplicity of $-2Kp$ be m and the multiplicity of 0 be $m - 4$. Then we have$m.(-2Kp) + (4 - m)\, 0 = r = -2Kp$ [by (4.6) and (4.12)].

Since $K \neq 0$, $p \neq 0$, we have $m = 1$. Thus the multiplicity of the eigenvalue $-2Kp$ is 1 and that of the eigenvalue 0 is 3. Hence the Segre characteristic[12] of S is [(111), 1]. This leads to the following theorem:

Theorem 3.4: In a perfect fluid $(PCPRS)_4$ space-time having Einstein equation without cosmological constant, the Segre characteristic of S is [(111), 1].

It is known that the energy and the force equations[7] for a perfect fluid are as follows:

$U\sigma = -(\sigma + p) divU$ and ...(4.20)

$(\sigma + p)\nabla_U U = -grad\, p - (Up)U.$ ... (4.21)

Since by (2.7) r is constant , it follows from (4.6) and (4.12) that σ and p are both constants. Hence we get from (4.20) and (4.21)

$Div\, U = 0$ and ... (4.22)

$\nabla_U U = 0.$ ...(4.23)

But div U represents the expansion scalar and ∇_UU represents the acceleration vector. In view of these it follows from (4.22) and (4.23) that the fluid has vanishing expansion scalar and vanishing acceleration vector. Hence we can state the following theorem:

Theorem 3.5: In a perfect fluid $(PCPRS)_4$ space-time having Einstein equation without cosmological constant, the fluid has vanishing expansion scalar and vanishing acceleration vector.

5. Conclusion

In section 2 we have considered Pseudo cyclic parallel Ricci symmetric Riemannian manifold It is shown that in a $(PCPRS)_n$ with associated vector field U defined by (3.3), the critical points of the energy f of the vector field U are either zeros of f or zeros of $a + 2f$. It is also shown that if the energy f of the vector field U is constant, then the integral curves of U are geodesics. In section 3 we have considered semi Riemannian general relativistic space-time. It is shown that in a perfect fluid $(PCPRS)_4$ space-time having Einstein equation without cosmological constant, if the energy momentum tensor T of the space- time obeys the time like convergence condition , then the space-time under consideration contains pure matter and the fluid has vanishing expansion scalar and vanishing acceleration vector.

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A Parameter Reduction Technique For Prediction of Pre-monsoon Thunderstorms Based On Fuzzy Multidimensional Degree of Compatibility

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[**Abstract:** In the present study, an attempt is made to propose a new parameter reduction technique for weather forecasting at Kolkata (22.53º N, 88.33º E), India, during the pre-monsoon season (March, April and May). The newly suggested technique is based on fuzzy multidimensional degree of compatibility. It can handle inherent non-linearity in a physical phenomenon. It is interesting to note that for the prediction of weather for next 12 hours based on Radio/Rawin Sonde observation at 1200 UTC of a day, the technique is better than any previous technique, although the previous techniques are however almost equally suitable to predict the weather of the next 12 hours based on Radio/ Rawin Sonde observation at 0000 UTC. The main objective of the study is to reduce the number of parameters without losing any important information for predicting the future situation. It is interesting to note that the methodology suggested in the study helps reduce the number of the parameters from 20 to 8 and 12 respectively for two different situation , fair weather and convective development for morning and evening to furnish almost 70% correct results. The degrees of compatibility are defined using a training data set for the period 1985-1996 and validated for the period 1997-1999].

Keywords: Convective development, forward selection rule, fuzzy multidimensional compatibility method, instability.

1. Introduction

Prediction of any atmospheric phenomenon is always of ultimate interest not only to the weather forecasters but to the common people as well. Specially, in recent years, there has been growing interest in the prediction of pre-monsoon convective developments (CD), not because of the possible hazards caused by them, but for their beneficial effects too like cooling due to rain during the hot summer days.

The convective developments occurring during March, April and May in Kolkata, India are termed as premonsoon thunderstorms. Different statistical techniques, like the principal component analysis (PCA), linear discriminant analysis (LDA), cluster analysis technique were applied by previous workers to identify the significant parameters for the occurrence of pre-monsoon thunderstorms (TS) in Kolkata. The linear discriminant analysis (LDA) technique alone as well as in conjunction with principal component analysis (PCA) could be successfully applied to a set of 20 parameters to reduce the dimension of the data matrix and predict the pre-monsoon thunderstorms for Kolkata.[1,2,3] Convective developments are generally favored by convective instability, abundant moisture at lower levels, strong wind shear, and a dynamical lifting mechanism that can release the instability.[4] Also, the vertical shear of the environmental winds has to match the value of the convective instability for proper development of a large convective cloud.[5] The presence of conditional instability is an essential criterion for supporting electrification and lightning.[6] In addition to the parameters mentioned, two other parameters, viz. $(\theta es - \theta e)$ and $(P - PLCL)$ are also present, where θes and θe denote the saturated equivalent potential temperature and equivalent potential temperature respectively. P is a level pressure and $PLCL$ is the pressure at the corresponding lifting condensation level.

The thermodynamic parameter $(\theta es - \theta e)$ was introduced by Betts[7] as a measure of the unsaturation of the atmosphere. PLCL for the surface parcel was considered as the cloud base[8] and hence (PPLCL) is taken as a forcing factor for the saturation of a parcel. Cluster analysis and LDA technique[9] were utilized to describe a multivariate statistical model for forecasting anomalies of surface pressure present over Europe and North Atlantic. In another study, multiple linear regression[10] was compared with LDA for making hind casts and real time forecasts of north-east Brazil wet season rainfall using sea surface temperature. Though a number of attempts[11, 12] were made to establish empirical models for the prediction of atmospheric stability/instability, the work done on Kano[13] is perhaps the first successful attempt for tropical region. Another attempt was made to predict the occurrence of CD at Dhaka (Bangladesh) in terms of stability indices.[14]

Fuzzy set theory, was originally proposed by Zadeh [15] aimed at imitating the model of human thought process. The basic premises of bi-valued true-false Boolean notion are redefined here. In spite of strong resistance to fuzzy logic, many researchers started working in the field during 1965- 1975. During the first decade, many mathematical structures were fuzzified by generalizing the underlying sets to be fuzzy, i.e. the sets with no sharp boundaries. The 90s was an era of new computational paradigms. The applications of fuzzy set theory include studies in many fields, e.g. meteorology, biology and others.[16]

In 1995 Murtha[17] applied the fuzzy logic in operational meteorology. Yu and Tao[18] developed a fuzzy multi-objective function for rainfall-runoff model calibration in 2000. In 2002, Gomes and Casanovas[19] reported a case study of solar irradiance which involved fuzzy logic and meteorological variables. In 2003, Mackay[20] used fuzzy logic in automated parameterization of land surface process models. Chang et al.[21] applied fuzzy theory in genetic algorithm to interpolate precipitation. Mitra et al.[22] used rule-based fuzzy inference system for weather forecasting. Also Ma et al.[23] applied the same technique for the verification of meso-scale NWP forecasts. Hubbert et al.[24] developed a technique for real time identification and filtering using fuzzy logic. Dhanya and Kumar[25] used a fuzzy rule based modeling approach for the prediction of monsoon rainfall in India. In the year 2011 a comparison between LDA technique and fuzzy membership roster method for pre-monsoon weather forecasting was presented by S.Ghosh et al.[26] Recently Mohammad Iqbal et al predicted weather pattern using fuzzy rough clustering.[27]

A new computational technique based on fuzzy multidimensional degree of compatibility is proposed in the work. The technique suggested in the study helps one to select the most effective combination of significant parameters out of 20 to discriminate the two important situations, convective development and fair weather 12 hours ahead during the pre-monsoon season of Kolkata. The methodology and the corresponding results are discussed in detail in the respective sections.

2. Data

The primary data collected are utilized to calculate the necessary thermodynamic and dynamic parameter for the study. The fuzzy rules for forecasting the convective development at Kolkata are constructed utilizing all the available radiosonde data of 12 years(1985-1996) and for the validation of the technique, the data of 3 years(1997-1999) are used.

The parameters $(\theta es - \theta e)$, (P – PLCL), $\frac{\partial \theta es}{\partial z}, \frac{\partial \theta e}{\partial z}, \frac{\partial v}{\partial z}$ for five different layers are used to construct the proposed fuzzy LOGIC BASED rule to discriminate the situations.[28]

In the literature, O_i $(i = 1$ to 20) represent the following thermodynamic and dynamic parameters. The study is however confined upto 500 hPa, due to the importance of this layer mentioned by many previous researchers.

O_1 = $(\theta es - \theta e)$ at 1000 hPa level ; O_2 = (P-PLCL) at 1000 hPa level; $O_3 = \partial \theta es/\partial z$ at 1000-850 hPa layer ; $O_4 = \partial \theta e/\partial z$ at 1000-850 hPa layer; $O_5 = \partial v/\partial z$ at 1000 –850 hPa layer; O_6 = $(\theta es - \theta e)$ at 850 hPa level; O_7 = (PPLCL) at 850 hPa level; $O_8 = \partial \theta es/\partial z$ at 850-700 hPa layer; $O_9 = \partial \theta e/\partial z$ at 850-700 hPa layer; $O_{10} = \partial v/\partial z$ at 850-700 hPa layer; $O_{11} = (\theta es - \theta e)$ at 700 hPa level; O_{12} = (P-PLCL) at 700 hPa level; $O_{13} = \partial \theta es/\partial z$ at 700-600 hPa layer ; $O_{14} = \partial \theta e/\partial z$ at 700-600 hPa layer; $O_{15} = \partial v/\partial z$ at 700 –600 hPa layer; $O_{16} = (\theta es - \theta e)$ at 600 hPa level ; O_{17} = (P-PLCL) at 600 hPa level; $O_{18} = \partial \theta es/\partial z$ at

600-500 hPa layer; $O_{19} = \partial\theta e/\partial z$ at 600-500 hPa layer; $O_{20} = \partial v/\partial z$ at 600–500 hPa layer. It is worth mentioning that the values of $(\theta es - \theta e)$ and (P – PLCL) at the lower level of each layer have been treated as their respective values for that layer. Here, z stands for vertical height, $\partial\theta es/\partial z$ for conditional instability, $\partial\theta e/\partial z$ for convective instability and $\partial v/\partial z$ for the vertical shear of horizontal wind.

3. Objective of the study

The main objective of the study is to construct a computational technique, which is simpler and cost effective compared to other existing methods of pre-monsoon weather prediction. Not only that another aim is to construct a general rule for parameter reduction which can be applied to other fields too, because the fuzzy logic based rules are more flexible to handle the nonlinearity of any natural phenomenon.

4. Methodology

The present study considers separately the following four situations

1. Prediction of convective development from the data of 0000 UTC (Morning CD or MTS)
2. Prediction of fair-weather from the data of 0000 UTC (Morning FW or MNTS)
3. Prediction of convective development from the data of 1200 UTC (Evening CD or ETS)
4. Prediction of fair-weather from the data of 1200 UTC (Evening FW or ENTS).

All the above mentioned predictions are made for the next 12 hours from the time of observations (It is found by the previous workers that the pre monsoon atmosphere in Kolkata differs structurally in Morning and Evening.[1] For each situation the prediction is made on the basis of the multidimensional degree of compatibility described in the subsection 4.1 as it is well known that any atmospheric phenomenon is essentially complex and multivariate in nature. The forward selection rule as discussed in subsection 4.2 is applied to select the most effective combination of parameters for discriminating the two cases, convective development and fair weather. The combinations of different parameters are selected since it is well known that any atmospheric phenomenon is essentially complex and multivariate in nature. The degrees of fuzziness of the underlying pattern classes are computed using membership validity measures with the help of the formula 8 in subsection 4.3. Finally the basic features of the methodology are discussed in subsection 4.4. Some basic Fuzzy Logic based rules used in the study are described in the following section:

4.1 Multidimensional degree of compatibility and justification of the choice of membership function

It is well known that a membership function is so constructed that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. The set defined by such membership function is called a fuzzy set. Let S denote the universal set of the parameters. Then the membership functions μ_X and μ_Y by which the fuzzy sets X and Y are defined have the forms: $\mu_X : S \rightarrow [0,1]$, $\mu_Y : S \rightarrow [0,1]$ where [0,1] denotes the interval of real numbers from 0 to 1, inclusive.[29] In the study the membership functions for the fuzzy sets of parameters are chosen to be Gaussian Membership Function. The justification for this choice is given at the end of this section.

Here we consider the two groups Y and X which are the two standard pattern classes for convective development and fair weather respectively. The sets Y and X are termed as fuzzy sets since it is difficult to identify sharp boundaries between these two sets so far the parameters convective instability, conditional instability and vertical shear are concerned. The degrees of compatibility of a parameter, O_i ($i = 1$ to 20) with the standard pattern classes, Y and X are computed on the basis of Gaussian membership function as follows:

$$A_Y(O_i) = exp\left\{-\frac{(O_i - m_{iCD})^2}{(\sigma_{ii}^2)}\right\}, i = 1\ to\ 20 \qquad \ldots(1)$$

$$A_X(O_i) = exp\left\{-\frac{(O_i - m_{iCD})^2}{(\sigma_{ii}^2)}\right\}, i = 1\ to\ 20 \qquad \ldots(2)$$

m_{iFW} and σ_{iFW}

where:

m_i : mean of the ith parameter of CD days (123 days for morning and 165 days for evening).

σ_{iCD} : standard deviation of the *i*th parameter of CD days (123 days for morning and 165 days f or evening).

m_{iFW} : mean of the *i*th parameter of FW days (280 days for morning and 201 days for evening).

σ_{iFW} : standard deviation of the *i*th parameter of FW days (280 days for morning and 201 days for evening).

In the present study, the range of values of the degree of compatibility is the interval (0,1).Here, the Gaussian function has two parameters m and σ, such that

$$\text{Gaussian}\,(O_1, m_1, \sigma_1) = \exp\left\{-\frac{(O_1 - m_1)^2}{(\sigma_1^2)}\right\} \quad \ldots(3)$$

where m_1 and σ_1 denote the center and width of the values of O_1 respectively. Since, the numerical values of the selective parameters are not scattered, the respective means of the thermodynamic and dynamic parameters represent the two patterns in a reliable way.

Finally, the degrees of compatibility of a day (i.e. a relevant pattern) defined by O = (O_1 , … … . , O_2) with the two standard pattern classes, Y and X are constructed as follows:

$$A_Y(O) = \prod_1^2 A_Y(O_i), i = 1 \text{ to } 20 \quad \ldots(4)$$

$$A_X(O) = \prod_1^2 A_X(O_i), i = 1 \text{ to } 20 \quad \ldots(5)$$

If, now, an unknown pattern or a day, say $U = (u_1,\ u_2, \ldots, u_{20})$ is given, where u_i is the quantified value associated with the ith parameter of the pattern, then the degrees of compatibility of U with the standard patterns, Y and X, denoted by A_Y(U) and A_X(U) respectively, are computed as follows:

$$A_Y(U) = \prod_1^2 A_Y(U_i), i = 1 \text{ to } 20 \quad \ldots(6)$$

$$A_X(U) = \prod_1^2 A_X(U_i), i = 1 \text{ to } 20 \quad \ldots(7)$$

An unknown pattern or a day, U is classified by the larger value of A_Y (U) or A_X(U), i.e. if A_Y (U) > A_X(U), then there is a possibility for U to be more of the pattern Y than of the pattern X for next 12 hours. So it may be predicted that U is expected to be a day with convective development for next 12 hours.[30]

Let us now justify the choice of membership function.

Justification for choice of membership function:

It is worth mentioning that there is no sound principle yet for guiding the choice of membership function or degree of compatibility. It is well known that there does not exist any standard method yet to choose a membership function. But, Gaussian membership function has been selected here because of the following reasons:

(i) Since some of the parameters are found to follow Gaussian distribution and usually the physical parameters are assumed to be Gaussian or quasi Gaussian in nature, for each parameter, the Gaussian membership function has been chosen to construct the one dimensional or univariate degree of compatibility.

(ii) Gaussian membership functions are continuously differentiable as well as parameterizable.

(iii) Gaussian membership functions are factorizable. Hence, we may synthesize a multi dimensional or multivariate degree of compatibility as the product of one dimensional or univariate degree of compatibility.

That is why the product forms have been used in the relations (6) and (7) to handle the nonlinearity . [10] This product form of membership functions is used already by Dhanya and Kumar.[23]

The nature of the membership function for an individual parameter $\theta es - \theta e$ is shown in the following graph as an example:

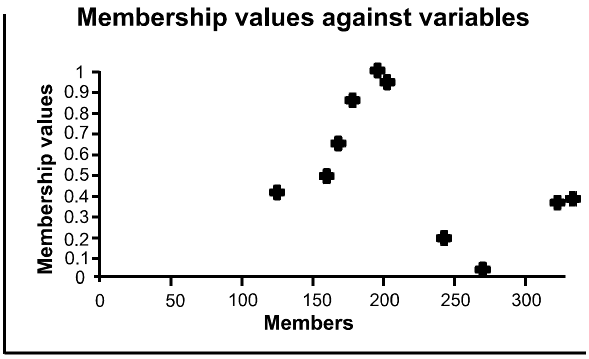


Figure 1

Graph plotting membership values v/s ($\theta es - \theta e$)

4.2 Forward Selection Rule for structure specification:

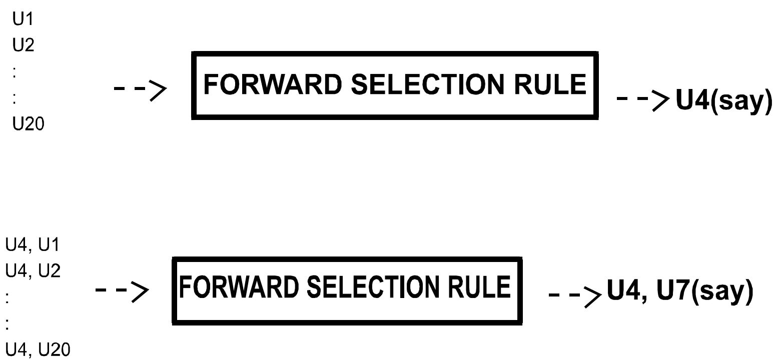
Step 1: Each of the twenty parameters is tested using the selection rule mentioned in the above section and that parameter is selected which produces maximum number of correct results for the data set used for validation.

Step 2: The combination of the remaining parameters with the selected one from Step 1 are tested to select the best combination (of two parameters).

Step 3: Proceeding as above the best combination of three parameters is selected. The process is repeated for all possible combinations of the twenty parameters.

This part of methodology is described with the help of the following block diagram:

Block Diagram



……and so

4.3 Degrees of fuzziness

In order to measure the degree of fuzziness of the pattern classes X and Y, the membership based validity measures, named as the partition coefficients[31] are computed separately for the four conditions as follows:

$$\left.\begin{aligned} V_{\text{PCMFW}} &= \left[\tfrac{1}{84}\right]\sum\left[A_X(u_j)+A_Y(u_j)\right] \\ V_{\text{PCMCD}} &= \left[\tfrac{1}{44}\right]\sum\left[A_X(u_j)+A_Y(u_j)\right] \\ V_{\text{PCEFW}} &= \left[\tfrac{1}{65}\right]\sum\left[A_X(u_j)+A_Y(u_j)\right] \\ V_{\text{PCECD}} &= \left[\tfrac{1}{53}\right]\sum\left[A_X(u_j)+A_Y(u_j)\right] \end{aligned}\right\} \quad \dots (8)$$

where u_j is the jth parameter for an unknown pattern or day, belonging to the dataset used for validation.

4.4 Basic Features of the Methodology

The present work includes the following three main stages:

Stage I: First the study is performed with all the 20 parameters using Forward Selection Rule described above.

For selecting single parameter equation (1) and (2) are used whereas for the selection of more than one parameters equations (4) and (5) are used. The corresponding results are presented with the combinations of all the twenty parameters in Table I. Since in this stage no common combinations of the parameters can be selected to discriminate the two situation under study so we proceed to Stage II.

Stage II: The combinations are considered first, which produce at least 75% correct result in each case. Further from the newly considered combinations, only the combinations of the physical parameters which are common for describing the situations in morning and evening are taken into account. The multidimensional degrees of compatibility of these new combinations are computed as before using equations (6) and (7).

Stage III: The membership validity measures are computed using the formula (8).

Table I

Nature of Day	Total No. of days for Verification	No. of Parameters involved	No. of Correct Prediction	% of Correct Prediction
ETS	53	20	24	**45.28**
ENTS	65	20	24	**36.92**
MTS	44	20	21	**47.73**
MNTS	84	20	72	**85.71**

Summary of the results from Stage I

5. Results and Discussion

Stage I: From our calculations it is found that the best combinations are obtained with O_5, O_{12}, O_{15}, O_1, O_{18}, O_4, O_{20}, O_{13}, O_{16}, O_6 for ETS.

Similarly we get the best combinations as O_8, O_3, O_{13}, O_4, O_{11}, O_{10}, O_{14}, O_9 for ENTS; O_{13}, O_{19}, O_{10}, O_{20}, O_8, O_4, O_{15}, O_{18}, O_{14}, O_5, O_9 for MTS; O_{19}, O_{13}, O_{10}, O_{15}, O_{18}, O_{20}, O_3, O_2, O_8 for MNTS respectively.

It is clear that there exist no combinations with common parameter to discriminate the situations either for Morning or Evening.
It may also be noticed from Table I that the average percentage of correct prediction with all the twenty parameters is not satisfactory except in MNTS.

Hence the Stage II is performed to select effective combinations with reduced number of parameters which may discriminate the situations, fair weather and convective development for morning and evening.

Stage II: The selection criterion described in subsection 4.4 helps reduce the number of selected parameters in each case, such that a combination of 12 common parameters for morning and a combination of 8 common parameters for evening are obtained to discriminate/ predict the situation.

The parameters thus chosen are as follows: O_1 = (θes – θe) at 1000 hPa level ; O_2 = (P-PLCL) at 1000 hPa level; O_4 = ∂θe/∂z at 1000-850 hPa layer; O_5 = ∂v/∂z at 1000 –850 hPa layer; O_8 =∂θes /∂z at 850-700 hPa layer; O_{10} = ∂v/∂z at 850-700 hPa layer; O_{13} =∂θes/∂z at 700-600 hPa layer ; O_{14} = ∂θe/∂z at 700-600 hPa layer; O_{15} =∂v/∂z at 700 – 600 hPa layer; O_{18}= ∂θes/∂z at 600-500 hPa layer; O_{19} =∂θe/∂z at 600-500 hPa layer; O_{20} = ∂v/∂z at 600–500 hPa layer for the morning are selected.

O_1 = (θes – θe) at 1000 hPa level ; O_3=∂θes/∂z at 1000-850 hPa layer ; O_4 = ∂θe/∂z at 1000-850 hPa layer; O_{11} = (θes – θe) at 700 hPa level; O_{12} = (P-PLCL) at 700 hPa level; O_{13} =∂θes/∂z at 700-600 hPa layer ; O_{15} = ∂v/∂z at 700 –600 hPa layer; O_{20} =∂v/∂z at 600–500 hPa layer for the evening are selected.

The results of the categorical discrimination of an unknown day (U) belonging to the dataset (1997, 1998 and 1999) on the basis of the newly selected combinations are presented in Table II.

As seen from Table II the combinations with the reduced number of parameters can discriminate the situations almost successfully. The success rate is 75% to 80% in morning and it is 71% to 75% in evening so far these data sets are concerned.

Table II

Nature of Day	Total No. of days for Verification	No. of Parameters involved	No. of Correct Prediction	% of Correct Prediction
ETS	53	08	40	**75.5**
ENTS	65	08	46	**71**
MTS	44	12	33	**75**
MNTS	84	12	68	**80.9**

Categorical Discrimination of an Unknown Day(U)

Stage III: The membership validity measures for ETS, ENTS, MTS, MNTS are computed with the help of relation (8) and are presented in Table III.

Regarding the membership validity measures it may be stated that the classes X and Y are not hard since VPCMFW $\neq 1$, VPCMCD $\neq 1$, VPCEFW $\neq 1$ and VPCECD $\neq 1$.

Hence the fuzzy rule based technique suggested here, has a possibility for improvemen.[31]

Table III

Nature of Day	**Membership based validity measure (VPC)**
ETS V	0.110
MTS V	0.001
MTS V	0.001
MNTS V	0.180

Membership based validity measure

6. Conclusion

It is worth mentioning that all the above thermodynamic and dynamic parameters are physically important for the situations, but for the operational purpose or for parameterization it is better to select the most effective combination of a fewer number of variables.

The program used for the study is developed by the authors themselves. The observation made in the study indicates that the computationally simpler as well as cost effective technique developed by the authors may help one reduce the number of parameters to analyse pre-monsoon atmospheric situation of a place provided a proper training set is available. But there are two limitations of the study

(i) The entire technique is context bound.

(ii) The data set used here is not primary, but secondary and assumed to be reliable.

The first limitation is inherent with any fuzzy logic based method. But to improve the reliability of data, primary data should be used whenever possible.

Incidentally the computational technique is validated in the present work with atmospheric data of Kolkata, India. But it is expected to work in other fields too for parameter reduction. Not only that but proper choices of membership functions may improve the technique effectively.

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