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Black Hole: Retrospect and Prospect

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Abstract

This comprehensive review traces the theoretical and observational evolution of black hole physics from its 18th century origins to contemporary developments. We examine the foundational contributions of Michell, Schwarzschild, Raychaudhuri, Penrose, and Hawking, emphasizing their roles in establishing black holes as inevitable consequences of general relativity. Recent observational breakthroughs, including the first Event Horizon Telescope images and gravitational wave detections, have transformed black holes from theoretical curiosities into observable astrophysical objects. This paper discusses current research frontiers, including loop quantum gravity modifications, dark matter interactions, and the thermodynamic properties of black holes, while highlighting future prospects in this rapidly evolving field.

Keywords: Black Hole, General Relativity, Raychaudhuri Equation, Gravitational Wave,

1. Introduction

1.1 Early Theoretical Foundations

The concept of a black hole, referring to a region in space where even light cannot escape, was first introduced by John Michell in 1783. Michell's idea originated from his contemplation of a hypothetical method to determine a star's mass. According to Newton's corpuscular theory of light, which posited that light consists of tiny particles, Michell proposed that the reduced velocity of these light particles upon emergence from a star's surface could be attributed to gravitational attraction. He reasoned that by measuring the decrease in light speed emanating from a particular star, it would be feasible to estimate the star's mass.

Based on the principle that any object needs to surpass a specific critical velocity—the escape velocity—to break free from a star's gravitational pull, Michell deduced that light particles must possess sufficient velocity to become visible from the star. However, the escape velocity depends on both the star's size and mass. Consequently, Michell concluded that if a star's gravitational force

were so intense that its escape velocity surpassed the speed of light, light particles would be unable to escape from the star's surface.

Building upon Ole Rømer's 1676 discovery of the finite speed of light, John Michell calculated that if a star were over 500 times larger than the Sun with the same average density, its escape velocity would exceed the speed of light. Consequently, light would be unable to escape from such a massive object due to its intense gravitational force, rendering these bodies invisible—a prescient insight into what we now call black holes.

Although Einstein's theory of relativity later established that light maintains a constant speed as it travels through space regardless of local gravitational strength, thus invalidating Michell's specific methodology for determining stellar mass, his fundamental insight remained significant. He had identified the crucial concept that any object would remain invisible if its escape velocity surpassed the speed of light—an observation that proved pivotal in the development of modern black hole theory.

2. Black Holes in Einstein's General Relativity

2.1 The Schwarzschild Solution

While the concept that gravity can become immensely powerful, preventing light from escaping, is inherent in Newtonian gravity theory, the true significance of black holes only captured physicists' attention when they recognized that black holes are inevitable consequences of general relativity. General relativity, which encompasses a set of equations describing the curvature of spacetime, is commonly known as Einstein's Field Equations.

In 1916, Karl Schwarzschild made the remarkable discovery that black holes could be exact solutions to Einstein's Field Equations [22], leading to their alternative designation as "Schwarzschild Solutions." The Schwarzschild solution primarily pertains to the exterior region of a spherically symmetric matter distribution. However, an intriguing question arises when considering what occurs at the surface of the matter distribution as it becomes increasingly compressed while maintaining a fixed mass.

Eventually, a critical radius is reached at which light is unable to escape from the star's surface or any region within it. This boundary surface is referred to as the event horizon of a black hole. The event horizon is characterized by a spherical region with a radius known as the Schwarzschild radius, which varies depending on the mass of the star. Consequently, each star possesses a unique Schwarzschild radius that determines the size of its corresponding black hole event horizon.

As matter continues to compress below this radius, the star undergoes gravitational collapse, eventually resulting in the formation of a singularity—a point of infinite density. For instance, the Schwarzschild radius for the Sun is approximately 3 kilometers. When approaching the horizon, scientists initially encountered mathematical quantities that appeared to tend toward infinity, leading to doubts regarding the physical validity of the Schwarzschild solution.

However, further investigation revealed that when physical quantities are measured by an observer approaching a black hole, none actually become infinitely large. The curvature of spacetime near a black hole varies inversely with the square of the mass. Through careful mathematical analysis employing coordinate transformations, the apparent "infinity" at the event horizon was shown to be a coordinate singularity rather than a physical singularity, successfully resolving these concerns.

2.2 Beyond Spherical Symmetry

Questions regarding the existence of black holes initially arose because the Schwarzschild solution was applicable only to spherically symmetric objects. However, this concern was definitively addressed through the groundbreaking singularity theorems developed by Roger Penrose and Stephen Hawking in the 1960s and 1970s [4, 5]. By rigorously examining Einstein's theory, they established that when a star lacks sufficient pressure and nuclear fuel to counteract its gravitational collapse, it will inevitably transform into a black hole, regardless of asymmetries in its structure.

Recent Development (2024-2025): Advanced numerical simulations using supercomputers have now confirmed that even highly asymmetric collapsing stars inevitably form black holes with remarkably spherical event horizons, vindicating the early theoretical predictions. These simulations, published in *Physical Review Letters* (2024), demonstrate the "cosmic censorship" hypothesis in action, showing how nature appears to hide naked singularities behind event horizons.

3. The Raychaudhuri Equation

3.1 Conceptual Foundation

According to Newtonian gravitational theory, if the distribution of matter in space is both homogeneous and isotropic, the gravitational force at any given point should cancel out. The isotropy of space implies equal and opposite forces in all directions, resulting in their vector sum being zero. Homogeneity further enforces this cancellation at all points in space. Consequently, in a perfectly homogeneous and isotropic universe, there would be no net gravitational force acting on any object.

This conclusion appears problematic, considering that matter density is non-zero, and we have Poisson's equation to account for gravitational effects. This peculiar result arises from treating gravity as a vector force. To resolve this fundamental issue, it becomes essential to consider gravity as a tensor force—a key insight of general relativity that treats gravity as spacetime curvature rather than a conventional force.

3.2 Cosmological Context

Amal Kumar Raychaudhuri had a profound impact on the field of cosmological modeling [1]. The exploration of the universe's large-scale properties using mathematical models gained significant momentum in 1917 when Einstein attempted to describe the universe as a solution to his field equations, initially considering the universe to be static.

The Einstein field equations are expressed as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = -(8\pi G/c^4)T_{\mu\nu} \quad (1)$$

These equations involve tensor quantities representing spacetime curvature on the left-hand side, while the physical properties of matter and energy are introduced through the energy-momentum tensor on the right-hand side. The term with the constant ' λ ' is known as the cosmological constant, originally introduced by Einstein to allow for a static universe.

Solving this set of ten nonlinear partial differential equations in four spacetime variables is extraordinarily challenging, and finding a complete general solution is practically impossible. Therefore, Einstein introduced simplifying assumptions about the universe to make the problem more tractable.

Later, Friedmann and Lemaître (1920s) developed theoretical cosmological models also derived from Einstein's general relativity. They made certain fundamental assumptions: the universe adheres to the cosmological principle (homogeneity and isotropy on large scales) and Weyl's postulate (galaxies follow non-intersecting timelike geodesics). Additionally, they assumed that the motion of matter on a large scale follows a highly regular pattern, allowing the use of comoving coordinates to describe space and time.

In their model, the worldlines of galaxies form a bundle of non-intersecting geodesics, with a system of spacelike hypersurfaces orthogonal to them. These hypersurfaces define cosmic time " t ," possessing the global property that at constant " t ," the space in the universe is both homogeneous and isotropic. Under these assumptions, the line element describing the spacetime geometry of the large-scale universe became known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = c^2 dt^2 - S^2(t)[dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (2)$$

Here, " $S(t)$ " is the scale factor describing the expansion of the universe, and the curvature parameter " k " can take values $+1$, 0 , or -1 , corresponding to positively curved (closed), flat, or negatively curved (open) spatial geometries, respectively.

3.3 The Singularity Problem

All Friedmann-Lemaître models describe an expanding universe, consistent with Edwin Hubble's astronomical observations. By the 1950s, it became evident that all models of this type have an epoch at which " $S(t)$ " becomes zero—the Big Bang singularity at $t=0$.

The Big Bang epoch holds a unique position in cosmology. For some, it represents the moment of creation. Prior to this epoch, the universe did not exist, but after the Big Bang, it began expanding at an incredibly rapid rate. The density and temperature were thought to be infinite at that moment. However, others view this as indicating a breakdown in our mathematical models. The notion of an infinite universe contained within zero volume at infinite temperature suggests that the models

used—such as the FLRW metric and Einstein's field equations—might break down at such extreme conditions.

To address this issue, some cosmologists sought ways to eliminate the singularity while maintaining faith in Einstein's field equations. One approach considered the possibility of the universe having angular momentum (rotation). In a rotating frame of reference, centrifugal forces come into play, potentially preventing the complete collapse of matter. This led some cosmologists to believe that rotating models might avoid singularities.

The first notable rotating model was proposed by Kurt Gödel in 1949. While mathematically intriguing, the Gödel universe lacks connection with observational reality. While researchers explored specific models, Raychaudhuri (1955) made an important general contribution with an equation that now bears his name [1, 2].

3.4 The Raychaudhuri Equation

The Raychaudhuri equation is [1, 2]:

$$d\Theta/ds + \Theta^2/3 = R_{\mu\nu}u^\mu u^\nu + 2\omega^2 - 2\sigma^2 + [du^k/ds] \quad (3)$$

where:

u^μ is the four-velocity vector of particles assumed to form a congruence of worldlines

Θ is the expansion scalar, measuring the divergence of the flow

ω measures the vorticity (rotation) of the congruence

σ measures the shear (distortion) of the congruence

d/ds denotes the rate of change along a worldline

du^k/ds is the four-acceleration

$R_{\mu\nu}$ is the Ricci tensor from Einstein's field equations

The singularity occurs when the expansion function Θ diverges to positive or negative infinity.

3.5 Physical Significance

The Raychaudhuri equation holds profound significance for several reasons [1, 2]. The first term on the right-hand side represents the gravitational effect of the universe's contents. Through Einstein's equations, this term is generally negative (assuming normal matter satisfies energy conditions), indicating gravity's tendency to cause focusing and potentially lead to singularities. The second term, arising from rotation (vorticity), has a positive sign and works to prevent singularity formation. The shear term has a negative sign, promoting focusing toward a singularity.

The equation leaves the outcome tantalizingly uncertain, but more importantly, it clearly distinguishes and identifies the dynamical effects from various sources controlling the universe's evolution. The Raychaudhuri equation laid the essential groundwork for subsequent research into the singularity problem, culminating in the celebrated singularity theorems of Penrose and Hawking [4, 5].

4. The Penrose Singularity Theorem

4.1 Historical Context

In 1915, Einstein's general relativity replaced Newtonian gravity, revealing that extremely dense mass could create a region where time effectively stops at the "event horizon." Within this boundary, all matter, light, and spacetime itself would be drawn inward toward a central point where gravity becomes infinitely strong—a singularity. However, the concept of physical infinities raises skepticism among physicists. Even Einstein questioned whether black holes could exist in the real universe and whether they would harbor singularities.

The Schwarzschild solution [22] provided a mathematical description of a black hole but did not explain how matter could become sufficiently dense to create an event horizon, nor whether such dense matter would inevitably collapse to a singularity. It merely demonstrated that once such a collapse occurred, the resulting black hole would be stable.

Oppenheimer and Snyder (1939) [24] demonstrated that a perfectly spherical, homogeneous ball of pressureless dust could collapse to form a Schwarzschild black hole, but only if all particles moved directly toward each other in a perfect radial infall, ultimately converging at the center. However, it was commonly believed that realistic objects could not collapse into a perfect singularity, as even slight deviations from perfect spherical symmetry would cause particles to miss each other, potentially preventing black hole formation. In such cases, a collapsing star might undergo a violent rebound rather than forming a black hole.

Roy Kerr (1963) [23] achieved a significant breakthrough by discovering an exact solution describing rotating black holes within Einstein's theory. He found that the central point of infinite gravity would be deformed into a ring singularity due to rotation, though it still represented a singularity. Despite this development, the Kerr solution retained high symmetry, leading to continuing doubts about whether such objects could exist in nature.

4.2 Penrose's Revolutionary Approach

In 1965, Roger Penrose, then at Cambridge, demonstrated that black hole singularities are inevitable within Einstein's theory under very general conditions [4, 5]. His goal was to prove that both an event horizon and a singularity would form for any sufficiently compact distribution of matter, regardless of asymmetries or irregularities, as long as matter satisfied reasonable energy conditions. He achieved this through a global geometric approach, showing that spacetime geodesics reach an endpoint inside a black hole.

In general relativity, spacetime structure is characterized by geodesics—curves representing the paths of freely falling particles and light rays. The paths followed by light rays are called null geodesics. These null geodesics tend to converge (focus) when passing through any gravitational field due to the attractive nature of gravity, a phenomenon known as gravitational lensing. Regular matter satisfying standard energy conditions always produces such focusing.

Penrose introduced the crucial concept of a "trapped surface" [4]—a closed surface where null geodesics attempting to move outward from the surface are actually forced to converge inward. Inside a black hole, the extreme spacetime curvature causes even outward-directed light rays to bend back inward. Any closed surface possessing this property is a trapped surface, and its presence indicates that gravitational collapse has proceeded beyond the point of no return.

4.3 Statement of the Theorem

The 1965 Penrose Singularity Theorem [4, 5] requires the following assumptions:

1. A 4-dimensional Lorentzian manifold with a metric g of class C^2 and causal orientation
2. A non-compact C^3 Cauchy hypersurface Σ
3. A closed future-trapped surface
4. $R_{\mu\nu}k^\mu k^\nu \geq 0$ for all null vectors k^μ (null convergence condition)
5. All null geodesics are complete toward the future of Σ

The theorem concludes that these assumptions are contradictory—if conditions 1-4 hold, then condition 5 must fail. This means null geodesics must be incomplete, indicating the presence of a singularity.

4.4 Physical Interpretation

The primary challenge with singularities is that they represent a breakdown of spacetime itself, making them fundamentally different from ordinary spacetime points. However, incomplete geodesics—curves that cannot be extended beyond a finite parameter value—provide valuable indicators and remain well-defined within spacetime. The work of Hawking and Geroch established geodesic incompleteness as the standard characterization of spacetime singularities [5].

Regarding dimensionality, while Penrose initially proved the theorem for four dimensions, it applies to any dimension $n \geq 3$. A closed trapped surface represents a compact submanifold of codimension 2, while a Cauchy hypersurface has codimension 1.

The convergence condition (null energy condition) addresses the attractive nature of gravity. This curvature condition is derived from Raychaudhuri's equation [1, 2] and ensures that gravity focuses causal geodesics. When expressed as $R_{\mu\nu}k^\mu k^\nu \geq 0$ for null vectors k^μ , this is the "null energy condition."

In general relativity, the Ricci tensor relates to the energy-momentum tensor through Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu} \quad (4)$$

The singularity theorems provide compelling evidence that understanding gravitational collapse and the very early universe requires physics beyond classical general relativity. Whether resolution requires a quantum theory of gravity remains uncertain, but modifications to Einstein's theory or violations of classical energy conditions appear necessary. The fundamental question of extending incomplete spacetimes requires decisions about when and how such extensions should be performed.

Nobel Recognition (2020): Roger Penrose was awarded the Nobel Prize in Physics in 2020 "for the discovery that black hole formation is a robust prediction of the general theory of relativity," explicitly recognizing the importance of his 1965 singularity theorem in establishing black holes as physical realities rather than mathematical curiosities.

5. Hawking Radiation

5.1 Theoretical Foundation

In 1974-1975, Stephen Hawking revolutionized black hole physics by demonstrating that black holes are not entirely black but emit thermal radiation [3]. His approach combined general relativity and quantum mechanics in a novel way. According to quantum field theory, empty space is not truly empty but filled with constant quantum activity. Virtual particle-antiparticle pairs spontaneously appear and annihilate throughout the vacuum, temporarily borrowing energy from it.

Near a black hole's event horizon, these quantum fluctuations take on special significance. When a virtual pair forms near the horizon, one particle may fall into the black hole while its partner escapes to infinity. The escaping particle becomes real, appearing as radiation emitted by the black hole. By energy conservation, the black hole must lose mass equivalent to the energy of the emitted particle, causing it to gradually evaporate [3].

5.2 Quantum Fields in Curved Spacetime

Space is permeated by quantum fields oscillating at various frequencies. Real particles represent excited states of these fields. Even in vacuum, the underlying vibrational modes exist. Due to quantum uncertainty, these modes fluctuate in energy, leading to the concept of virtual particles as computational tools for understanding quantum field behavior.

A quantum field can exhibit both positive and negative frequency modes, with negative frequencies corresponding to modes apparently traveling backward in time and associated with antimatter. In vacuum state, a delicate balance exists between positive and negative frequency modes—an equilibrium between matter and antimatter. Virtual particles continuously annihilate, preventing actual particle creation. This description holds in flat spacetime.

However, in regions with strong spacetime curvature, such as near event horizons, this balance is disturbed. Horizons effectively block certain quantum field modes, disrupting the equilibrium.

Hawking recognized that black holes, with their extreme spacetime curvature, would profoundly affect nearby quantum fields [3].

5.3 The Bogoliubov Transformation

To address this quantum-gravitational effect, Hawking employed the Bogoliubov transformation [3]—a mathematical technique allowing him to approximate curved spacetime's impact on quantum fields. This approach connects the flat-space vacuum far from the black hole to the quantum state near the horizon, describing how spacetime curvature mixes positive and negative frequency modes.

Hawking's calculation revealed that certain quantum field modes are scattered by the forming black hole's gravitational field and become trapped behind the event horizon. Other modes escape unaffected. Loss of fundamental modes means the vacuum state must be reconstructed from remaining modes. This distorted vacuum appears to contain particles—the origin of Hawking radiation.

5.4 Temperature and Spectrum

Black holes preferentially scatter modes with wavelengths comparable to their size. Consequently, the resulting radiation consists of particles with wavelengths approximately equal to the event horizon size. The radiation's wavelength depends directly on the black hole's mass—more massive black holes produce longer-wavelength, lower-energy radiation.

Hawking calculated this radiation's frequency distribution [3], finding it precisely matches thermal blackbody radiation with temperature:

$$T_H = \hbar c^3 / (8\pi G M k_B) \quad (5)$$

where $\hbar$ is the reduced Planck constant, c is light speed, G is the gravitational constant, M is black hole mass, and k_B is Boltzmann's constant.

Massive black holes appear cold, emitting radiation slowly. Smaller black holes appear hotter, potentially emitting radiation explosively as they approach complete evaporation.

5.5 Observational Challenges

Hawking radiation is non-localized; its wavelengths scale with the entire event horizon. These are essentially de Broglie wavelengths, implying significant quantum uncertainty in particle locations. Therefore, Hawking radiation appears to emanate from the entire black hole, not specific horizon points.

Importantly, an observer freely falling through the event horizon perceives nothing unusual—space appears locally flat, vacuum appears normal. This radiation is observable only to distant observers. Generally, Hawking radiation consists of photons and massless particles. However,

Unruh radiation—a related phenomenon—can be detected when observing the horizon from a fixed distance above it.

Recent Developments (2024): Experimental analog systems using sound waves in flowing fluids (acoustic black holes) and light in nonlinear optical media have provided indirect evidence supporting Hawking's theoretical predictions [21]. While detecting actual Hawking radiation from astrophysical black holes remains beyond current technology due to its extreme faintness, these analog experiments demonstrate the fundamental quantum processes underlying the effect.

6. Angular Momentum and Charge in Black Holes

6.1 The First Law of Black Hole Mechanics

A rotating or electrically charged collapsing object forming a black hole will eventually settle into a Kerr-Newman solution (Kerr solution for uncharged rotating black holes [23], Reissner-Nordström for charged non-rotating black holes). When a wave packet interacts with a rotating, charged black hole, the change in the black hole's mass dM relates to changes in its area A , angular momentum J , and charge Q through the first law of black hole mechanics [25]:

$$dM = (\kappa/8\pi)dA + \Omega dJ + \phi dQ \quad (6)$$

where M is mass, J is angular momentum, Q is charge, Ω is the black hole's angular velocity at the horizon, ϕ is the electrostatic potential, and κ is surface gravity.

The ratio between energy, angular momentum, and charge fluxes is $\omega:m:e$ (where ω is wave frequency, m is azimuthal quantum number, e is charge). Consequently, changes in the black hole's properties maintain this ratio:

$$dM(1 - \Omega m/\omega - e\phi/\omega) = (\kappa/8\pi)dA \quad (7)$$

6.2 Superradiance

For wave packets satisfying weak energy conditions (non-negative local energy density), the area change dA must be non-negative. If $\omega < m\Omega + e\phi$, then dM must be negative—the black hole loses energy to the wave packet, which scatters with increased amplitude. This phenomenon is called "superradiance," where the black hole acts as an amplifier, extracting rotational energy from the black hole.

Superradiance has profound implications for black hole physics and has been proposed as a mechanism for various astrophysical phenomena, including the formation of "black hole bombs" when reflecting boundaries trap superradiant modes.

7. Geometry of Rotating and Magnetized Black Holes

7.1 Embedding in Three-Dimensional Space

According to Smarr's research, rotating black holes exhibit similarities to rotating material bodies—they become oblate (flattened) as they spin faster. Smarr demonstrated that a Kerr black hole cannot be globally and isometrically embedded in three-dimensional Euclidean space R^3 if its specific angular momentum (angular momentum per unit mass) exceeds $\sqrt{3}/2$.

However, when a magnetic field of strength B is introduced satisfying $(2-\sqrt{3}) \leq B^2 M^2 \leq (2+\sqrt{3})$, global isometric embedding in R^3 becomes possible for all angular momentum values [6]. This reveals the interplay between rotation and magnetic fields in determining black hole geometry.

7.2 Gaussian Curvature and Magnetic Fields

Wild and Kerns examined a Schwarzschild black hole placed in a uniform magnetic field, discovering that Gaussian curvature becomes negative at the equator when $B > 1/M$. They initially concluded that global isometric embedding in R^3 was impossible under these conditions.

However, Kulkarni and Dadhich subsequently demonstrated this inference was incorrect [6]. They proved that global isometric embedding conditions are independent of Gaussian curvature sign at specific points. Furthermore, rotation and magnetic fields counteract each other: negative curvature from rotation at poles can be compensated by appropriate magnetic fields.

Consequently, axial symmetry combined with negative Gaussian curvature at poles prevents global isometric embedding in R^3 . However, a Kerr black hole in a uniform magnetic field can be globally embedded in R^3 for all angular momentum values within a specific magnetic field strength range [6]. Therefore, Gaussian curvature at non-polar points does not preclude global isometric embedding.

8. Observational Evidence for Black Holes

8.1 X-Ray Binary Systems

X-ray binaries consist of a compact stellar remnant accreting material from a normal companion star. They offer exceptional opportunities to investigate stellar collapse in various environments, refining our understanding of Type Ia and Type II supernovae physics [7]. These compact remnants become visible through ongoing or sporadic X-ray emission from mass accretion. Observationally, they categorize into three types: pulsars, neutron stars, and black holes, each paired with companion donor stars spanning wide mass ranges.

Historically, X-ray binaries have been classified by donor star mass: low-mass X-ray binaries (LMXB) with donors $M < 1-2 M_{\odot}$, and high-mass X-ray binaries (HMXB) with more massive donors [7]. The observed X-ray activity depends on three key factors:

1. The compact star's magnetic field strength
2. Mass transfer rate from the donor star
3. Accretion disk heating by accretion luminosity

These variables explain why black hole remnants predominantly appear in transient LMXBs, neutron stars in persistent LMXBs, and pulsars frequently in HMXBs [7]. We have observed millisecond-period pulsars in transient LMXBs, regarded as crucial evidence in X-ray binary evolution where neutron stars are spun up through sustained accretion, ultimately becoming recycled millisecond pulsars.

8.2 Confirmed Black Hole Candidates

The majority of accreting black holes reside in transient LMXBs and intermediate-mass X-ray binaries (IMXBs), confirmed through detailed dynamical studies [7]. During quiescent phases, highly accurate mass measurements have been obtained using photometric and spectroscopic methods to detect companion stars. Notable examples include:

- i. **GRS 1915+105** (transient LMXB): One of the most massive stellar black holes known, with mass estimates around $12\text{--}14 M_{\odot}$
- ii. **V404 Cygni** (transient LMXB): Historical importance as an early confirmed black hole, mass $\sim 9 M_{\odot}$
- iii. **GRO J1655-40** (transient IMXB): Notable for high-velocity jets, mass $\sim 6.3 M_{\odot}$
- iv. **GX 339-4**: Constraints on black hole mass derived using the Bowen fluorescence technique

However, substantial uncertainties can affect these measurements due to systematic errors in determining binary inclination angles.

Confirmed black holes in HMXBs include [7]:

- i. **Cygnus X-1** (persistent): The first black hole candidate, mass $\sim 21 M_{\odot}$
- ii. **LMC X-1** (persistent): Located in the Large Magellanic Cloud
- iii. **LMC X-3** (persistent): Another LMC system
- iv. **M33 X-7** (persistent): One of the most massive stellar black holes discovered, $\sim 16 M_{\odot}$

The initial black hole discovery was Cygnus X-1, located in the constellation Cygnus within our Milky Way. MWC 656 represented the first instance where a black hole was identified orbiting a Be star (a rapidly rotating B-type star with emission lines). To determine black hole masses in X-ray binaries, scientists employ spectroscopic measurements of the optical companion's mass function and radial velocity curves, derived from dynamics of the surrounding gaseous disk [7].

Recent Discoveries (2023-2025): The Gaia mission has revolutionized black hole discovery through astrometric measurements. Gaia BH1 and Gaia BH2, discovered in 2022-2023, represent dormant black holes—systems with no active accretion—detected purely through their gravitational influence on companion stars. These discoveries suggest a large population of "invisible" black holes throughout the galaxy. As of 2024, over 20 stellar-mass black holes have been dynamically confirmed in our galaxy, with the total population estimated at 108-109 objects.

8.3 Space Velocities of Black Holes

Black holes originating from fallback onto proto-neutron stars can acquire significant velocity boosts, potentially leading to substantial space velocities. This velocity magnitude depends on whether natal kicks—asymmetric supernova explosions—impart similar momentum to both neutron stars and collapsing proto-neutron stars forming black holes [7].

Additionally, any supernova explosion in a binary system involving substantial mass loss imparts a velocity kick to the system, regardless of the compact object type formed. If more than half the total binary mass is ejected, the binary can be disrupted, imposing an upper limit on the "Blaauw kick." Notably, Blaauw kicks are directed within the binary orbital plane, whereas natal kicks may have arbitrary directions.

Evidence for velocity kicks in both neutron star and black hole LMXBs is observed in their relatively large scale heights above the galactic plane. Furthermore, evidence from LMXB distributions in early-type galaxies, where LMXBs extend beyond stellar light distribution, suggests kicked LMXB populations. Recent evidence for black holes in globular clusters indicates a population with low natal kicks, contrasting with high-velocity field black holes [7].

9. Direct Imaging: The Event Horizon Telescope

9.1 The First Black Hole Image (2019)

The initial black hole identification occurred with Cygnus X-1 in 1964 when sounding rockets detected celestial X-ray sources. However, the most revolutionary observational breakthrough came in April 2019, when the Event Horizon Telescope (EHT) collaboration successfully captured the first direct image of a supermassive black hole's shadow [12].

The photographed black hole resides at M87 galaxy's core, located 55 million light-years from Earth, with mass approximately $6.5 \text{ billion } M_{\odot}$ [12]. This achievement required coordinating eight radio telescopes across the globe, effectively creating an Earth-sized virtual telescope through Very Long Baseline Interferometry (VLBI). The image revealed a bright ring of emission surrounding a dark central shadow—the photon sphere where light orbits the black hole—with the shadow representing the event horizon's silhouette.

The asymmetric brightness in the ring results from relativistic beaming: material approaching Earth appears brighter due to Doppler boosting. The image's resolution reached approximately 20 microarcseconds, equivalent to photographing a DVD on the Moon from Earth [12].

9.2 Improved Imaging Techniques (2023)

In 2023, the EHT collaboration published an improved image of M87's black hole using Principal Component Interferometric Modeling (PRIMO) algorithm applied to the same 2017 dataset [13]. This advanced technique produced a sharper, more detailed image revealing finer structure in the emission ring and better constraining the black hole's properties.

The refined analysis confirmed the original mass estimate and provided tighter constraints on the black hole's spin. The persistent ring structure across multiple years of observations demonstrates

the stability of the accretion flow and validates general relativity predictions in this extreme gravitational regime [13].

Sagittarius A* Imaging (2022): In May 2022, the EHT collaboration revealed the first image of Sagittarius A* (Sgr A*), the supermassive black hole at our Milky Way's center [14]. With mass ~ 4 million $M_{\odot}$ and distance $\sim 26,000$ light-years, Sgr A* is much closer but smaller than M87*. Its rapid variability (matter orbits in minutes rather than weeks) required new imaging techniques averaging thousands of images. The resulting image showed similar ring structure to M87*, providing strong evidence that supermassive black holes share universal properties regardless of mass.

10. Gravitational Wave Astronomy

10.1 LIGO/Virgo Detections

The September 14, 2015 detection of gravitational waves from merging black holes (GW150914) by LIGO inaugurated gravitational wave astronomy [15]. This historic detection confirmed Einstein's century-old prediction and provided direct evidence for stellar-mass black hole binaries. The system consisted of black holes with masses $\sim 36 M_{\odot}$ and $\sim 29 M_{\odot}$, merging to form a $\sim 62 M_{\odot}$ black hole, with $\sim 3 M_{\odot} c^2$ radiated as gravitational waves.

Since then, the LIGO-Virgo-KAGRA collaboration has detected over 90 gravitational wave events, predominantly from black hole mergers, revealing:

- i. Black holes span masses from $\sim 5 M_{\odot}$ to over $100 M_{\odot}$
- ii. Intermediate-mass black holes (100 - $1000 M_{\odot}$) exist, as evidenced by GW190521
- iii. Black hole spins exhibit wide distributions, constraining formation scenarios
- iv. Merger rates indicate frequent black hole binary formation throughout cosmic history

10.2 Tests of General Relativity

Gravitational wave observations provide unprecedented tests of general relativity in strong-field, dynamical regimes. The observed waveforms match numerical relativity predictions with remarkable precision [15], confirming:

- i. Gravitational waves propagate at light speed (within 10^{-15} precision from GW170817)
- ii. Post-merger ringdown matches predictions for perturbed Kerr black holes
- iii. No evidence for violations of Lorentz invariance or graviton mass
- iv. Black holes are described by the no-hair theorem—completely characterized by mass and spin

Pulsar Timing Array Results (2023): On June 28, 2023, multiple pulsar timing array collaborations (NANOGrav, EPTA, PPTA, IPTA) announced the first evidence for a stochastic gravitational wave background at nanohertz frequencies [16]. This "gravitational wave hum" likely originates from countless supermassive black hole binary systems (10^6 - $10^{10} M_{\odot}$) throughout

cosmic history. Unlike LIGO's detection of individual merger events, this represents the collective murmur of the universe's most massive black hole systems [11].

11. Future Prospects and Open Questions

11.1 Quantum Gravity and Loop Quantum Gravity

As described in Einstein's general relativity, black holes are spacetime regions ending in "singularities"—points of infinite density where classical physics breaks down. However, Loop Quantum Gravity (LQG) offers an alternative perspective [8, 9], proposing that spacetime consists of discrete, indivisible units (quantum geometry) at the Planck scale ($\sim 10^{-35}$ meters).

In LQG, the classical singularity is replaced by a "quantum bounce" [8, 9]—as collapsing matter reaches Planck density ($\sim 10^{96}$ kg/m³), quantum geometric effects become dominant, preventing infinite compression. This suggests black holes might serve as bridges to other regions of spacetime or new universe regions, with the singularity replaced by a high-curvature but finite quantum region.

Recent theoretical work suggests observable signatures: modifications to Hawking radiation spectrum at late stages of evaporation, and potential quantum corrections to gravitational wave signals from black hole mergers [8]. However, these effects remain extremely small for astrophysical black holes, and compelling observational evidence awaits more sensitive instruments.

Recent Theoretical Progress (2024): New calculations using Loop Quantum Gravity predict that black holes might have finite lifetimes not through complete evaporation, but through transition to "white hole" remnants—objects that repel rather than attract matter [17]. Numerical simulations suggest this transition time could be extremely long for stellar-mass black holes (far exceeding the universe's age) but potentially observable for primordial black holes if they exist.

11.2 Dark Matter and Black Hole Interactions

Indirect evidence suggests significant dark matter concentrations near supermassive black holes. Dark matter particles, interacting gravitationally with black holes, can form dense "spikes" in the immediate vicinity, potentially affecting black hole dynamics and growth [10]. Recent theoretical work suggests:

- i. Dark matter annihilation near black holes could produce detectable gamma-ray signals
- ii. Dark matter accretion might dominate black hole growth in some environments
- iii. Gravitational wave signals from black hole mergers could carry imprints of surrounding dark matter distributions

The 2023 discovery that dark matter spikes around supermassive black hole binaries might contribute to the nanohertz gravitational wave background [10] opens new avenues for dark matter detection through gravitational wave astronomy.

11.3 Black Hole Thermodynamics and Information Paradox

Exploring black holes as thermodynamic objects with temperature and entropy reveals deep connections between gravity, quantum mechanics, and thermodynamics. The Bekenstein-Hawking entropy formula [25] $S = kc^3A/(4\hbar G)$ relates black hole entropy to horizon area, suggesting that information is somehow encoded on the horizon surface—the holographic principle.

However, Hawking radiation appears thermal, carrying no information about the matter that formed the black hole [3], leading to the information paradox: quantum mechanics demands information conservation, yet black hole evaporation seems to destroy information. Recent theoretical developments suggest:

- i. Information might be preserved through subtle correlations in Hawking radiation
- ii. Quantum entanglement between interior and exterior regions (ER=EPR conjecture) might resolve the paradox
- iii. Black hole horizons might be more complex than classical smooth surfaces, with quantum "hair" encoding information

Recent Breakthrough (2024): New calculations combining string theory, quantum information theory, and gravitational path integrals suggest that information does escape black holes, but encoded in extremely subtle correlations requiring measurements of unprecedented precision [18]. These "islands" in the entanglement structure appear essential for unitarity (information conservation) in quantum gravity. While experimental verification remains distant, this represents significant theoretical progress toward resolving the information paradox.

11.4 Primordial Black Holes

Primordial black holes (PBHs), potentially formed in the early universe from density fluctuations, offer fascinating possibilities. Unlike stellar black holes, PBHs could have any mass from microscopic to supermassive scales. They represent potential dark matter candidates, particularly in the asteroid-mass range (10^{17} - 10^{22} g) where constraints are weakest.

Recent investigations focus on:

- i. Gravitational lensing searches for PBHs through microlensing events
- ii. Gravitational wave observations of binary PBH mergers
- iii. Constraints from cosmic microwave background and large-scale structure
- iv. Potential role in seeding supermassive black hole formation

11.5 Testing General Relativity with Extreme Mass Ratio Inspirals

Future space-based gravitational wave detectors, particularly LISA (Laser Interferometer Space Antenna), will observe extreme mass ratio inspirals (EMRIs)—stellar-mass objects spiraling into supermassive black holes. These systems provide extraordinary laboratories for testing general relativity in strong fields, mapping spacetime geometry around supermassive black holes with unprecedented precision.

EMRIs will enable:

- i. Precision measurements of black hole spin and mass
- ii. Tests of the no-hair theorem at percent-level accuracy
- iii. Detection of environmental effects (accretion disks, dark matter halos)
- iv. Searches for deviations from Kerr geometry indicating new physics

11.6 Black Hole Accretion and Jet Physics

Understanding how black holes accrete material and launch powerful jets remains a major challenge. Recent advances include:

- i. High-resolution simulations of magnetohydrodynamic accretion flows
- ii. EHT observations revealing magnetic field structure near event horizons [12, 13]
- iii. Understanding how black hole spin energy powers relativistic jets
- iv. Connections between accretion states and jet launching/suppression

These investigations illuminate energy release mechanisms in quasars, active galactic nuclei, and gamma-ray bursts—some of the universe's most energetic phenomena.

11.7 Black Holes and Galaxy Evolution

The relationship between supermassive black holes and their host galaxies continues as a vibrant research frontier. Observations reveal tight correlations between black hole mass and galaxy properties (bulge mass, velocity dispersion), suggesting co-evolution. Key questions include:

- i. How do seed black holes form in the early universe?
- ii. What mechanisms enable rapid growth to billion solar mass scales by $z \sim 7$?
- iii. How does black hole feedback regulate star formation?
- iv. What role do black hole mergers play in hierarchical galaxy assembly?

JWST Discoveries (2023-2024): The James Webb Space Telescope has revolutionized early universe observations, detecting active supermassive black holes at redshifts $z > 10$, less than 500 million years after the Big Bang [19, 20]. These discoveries challenge formation scenarios, as growing black holes to billions of solar masses so quickly requires either massive seeds (10^4 - $10^5 M_\odot$) or sustained super-Eddington accretion. JWST observations suggest both mechanisms might operate, with direct collapse black holes providing massive seeds and accretion through dense gas streams enabling rapid growth.

11.8 Microscopic Black Holes and Particle Physics

Theoretical models with extra dimensions predict that microscopic black holes might be produced in high-energy particle collisions at facilities like the Large Hadron Collider (LHC). While no such events have been observed, searches continue and provide constraints on extra-dimensional theories. These hypothetical micro black holes would:

- i. Evaporate almost instantaneously via Hawking radiation [3]
- ii. Produce characteristic decay signatures in particle detectors
- iii. Reveal information about spacetime dimensionality and quantum gravity

11.9 Ethical and Philosophical Considerations

While manipulating or controlling black holes remains firmly in science fiction territory, philosophical questions arise:

- 1) What are the implications of information loss for determinism and causality?
- 2) How do black holes relate to fundamental questions about time's arrow and entropy?
- 3) What do black hole physics and holography reveal about reality's fundamental nature?
- 4) Could advanced civilizations utilize black hole energy (Penrose process, Hawking radiation)?

12. Conclusion

From Michell's 18th-century speculation to modern gravitational wave detections and direct imaging, black hole physics has evolved from mathematical curiosity to observational reality. The field stands at an exciting juncture where theoretical predictions meet experimental verification with increasing precision.

Recent breakthroughs—EHT images revealing event horizon structure [12, 13, 14], LIGO-Virgo-KAGRA detections mapping the black hole population [15], pulsar timing arrays detecting the gravitational wave background [16], and JWST observations of early supermassive black holes [19, 20]—demonstrate that black hole physics remains at the forefront of fundamental science.

Future developments promise even deeper insights. Loop quantum gravity and quantum information theory approach the information paradox from new angles [8, 9, 17, 18]. Next-generation gravitational wave detectors will probe black hole mergers with unprecedented sensitivity. Improved radio telescope networks will image black hole dynamics in real-time. Theoretical advances continue unifying quantum mechanics and gravity.

Black holes serve as nature's ultimate laboratories, where gravity, quantum mechanics, thermodynamics, and information theory intersect. Their study illuminates not only these exotic objects themselves but fundamental questions about spacetime, quantum theory, and the universe's origin and fate. As observational capabilities and theoretical understanding advance together, black hole physics will undoubtedly yield surprises, challenges, and profound insights into reality's deepest structure.

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Analysis Of Some Thermoluminescence Glow Curves Using Gaussian Functions

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Abstract

In the present work, rigorous analysis of Thermoluminescence (TL) curves have been carried out with Gaussian function in a systematic mathematical procedure. The results obtained thereof have been employed to extract the trapping parameters from different approaches such as peak shape method, inflection points method. An effort has been made to estimate the effective order of kinetics of TL peaks. It is observed that use of Gaussian function for glow curve deconvolution might lead to unusually large values of figure of merit. From a theoretical point of view, the present findings explicitly show the inconsistency among the values of trapping parameters determined in various ways which indicates the limitations of Gaussian functions in TL glow curve analysis.

Keywords: Thermoluminescence, glow curve analysis, Gaussian function, figure of merit, inflection points

1. Introduction:

Computerized glow curve deconvolution is a well-known and widely used technique for extracting the trapping parameters from a complex thermoluminescence (TL) glow curve. Many types of deconvolution functions are used in this method [1] and Gaussian function is one of them. Chandrashekhar and Gartia [2] have shown that in case of deconvolution of complex TL glow curves and extracting the trapping parameters thereof, use of Gaussian functions cannot be treated as a suitable measure as they yield physically unrealistic values for such parameters. However, the results reported in Ref. [2] on which the proposition was based are questionable. The methodology was also not quite explicit at certain points. For example, the equations used for fitting and the values of the fitting parameters are not given in Ref. [2] whereas these parameters are useful for subsequent studies and also for broad readership. The specific expressions for determination of frequency factor (s) was also not mentioned whereas, in literature, there

exist several phenomenological approaches to calculate 's' from TL curves. In this context, we have carried out TL analysis of the curves reported in Ref. [2] in an elaborate manner by using Gaussian functions for curve fitting and made several efforts to estimate the trapping parameters to check the mutual consistency among themselves. In this study, we have reanalyzed the glow curves of Ref. [2] using Gaussian fitting with a more reliable approach.

2. Methodology

2.1. TL curve generation

The general order kinetics (GOK) equation for TL intensity is given by [3]

$$I(T) = n_0 s e^{-E/kT} \left[1 + \frac{(b-1)s}{\beta} \int_{T_0}^T e^{-\frac{E}{kT'}} dT' \right]^{\frac{-b}{(b-1)}}, \text{ for } b \neq 1 \quad (1)$$

Here, n_0 = initial concentration of trapped electrons, E = activation energy, b = order of kinetics, s = effective frequency factor, T_0 = initial temperature, β = linear heating rate. In the limit $b \rightarrow 1$ equation (1) reduces to the first order kinetics equation [4]

$$I(T) = n_0 s \exp\left(-\frac{E}{kT}\right) \exp\left[-\frac{s}{\beta} \int_{T_0}^T e^{-E/kT'} dT'\right] \quad (2)$$

TL curves have been generated by using equation (1). The input parameters (E , s , b) for generation of TL peaks are all taken from general order kinetics results of Ref. [2].

2.2. Fitting with Gaussian functions

The TL curves are now fitted with Gaussian functions which are dependent on two parameters [2]. Here, for curve fitting, we use the following Gaussian function

$$I_g(T) = I_0 + \frac{A}{\sigma} \sqrt{\frac{2}{\pi}} \exp\left[-2\left(\frac{T-T_{mg}}{\sigma}\right)^2\right] \quad (3)$$

where $I_g(T)$ is the TL intensity at temperature T , I_0 = offset, A = area. The peak maxima occurs at the mean position (T_{mg}) and σ is the width at inflection points (T_i^- in the rising side and T_i^+ in the falling side of the curve). Hence,

$$\frac{d^2 I}{dT^2} = 0 \text{ at } T = T_i^\pm = T_{mg} \pm \frac{\sigma}{2} \quad (4)$$

The Gaussian fitting is carried out using Qti-Plot software where the Levenberg- Marquardt algorithm is adopted (tolerance = 0.0001). We have also estimated the figure of merit (FOM) for all the curves. Instead of using the expression for FOM used in Ref. [2], we have used the relation given by

$$\text{FOM} = \frac{\sum_i I(T) - Ig(T)}{\sum_i Ig(T)} \times 100\% \quad (5)$$

Here $I(T)$ is the TL intensity at temperature T according to equation (1). The χ^2 - values [6] for all the peaks are also estimated.

2.3. Estimation of trapping parameters

According to Gaussian distribution, the $\frac{1}{2}$ - intensity temperatures occur at

$$T_{1/2}^{\pm} = T_{mg} \pm \sigma \left[\frac{1}{2} \ln \frac{2(I_m - I_0)}{I_m - 2I_0} \right]^{\frac{1}{2}} \quad (6)$$

$T_{\frac{1}{2}}^{-}$ and $T_{\frac{1}{2}}^{+}$ are in the rising side and the falling side of the glow curve respectively. I_m = maximum intensity (at $T = T_{mg}$). Hence, the symmetry factor $\mu_g = \left(\frac{T_{\frac{1}{2}}^{+} - T_{mg}}{T_{\frac{1}{2}}^{+} - T_{\frac{1}{2}}^{-}} \right)$ is always 0.5.

In the next step, we have carried out the calculations of effective order of kinetics (b) . Using $\mu_g = 0.5$ and the values of T_{mg} , $T_{\frac{1}{2}}^{\pm}$, the average activation energies (E_c) for all the peaks are estimated using peak shape formula of Chen [7], as is done in Ref. [2]. These E_c values are employed to determine 'b' according to the equation of Kitis and Pagonis [1]

$$\mu_g = \left[\frac{b E_c}{E_c + 2kT_{mg}(b-1)} \right]^{\frac{-1}{b-1}} \quad (7)$$

Here we have employed the numerical technique adopted by Karmakar et al. [8].

The frequency factors (s) corresponding to each peak is determined in two different ways.

i) The approximated expression [3, 4] is used to estimate the frequency factor.

$$s = \exp \left(\frac{E_c}{kT_{mg}} \right) \quad (8)$$

ii) The maxima condition [3] of a general order TL peak (equation 1) is given by

$$\frac{(b-1)s}{\beta} \int_{T_0}^{T_{mg}} \exp\left(\frac{-E_c}{kT}\right) dT + 1 = \frac{sbkT_{mg}^2}{\beta E_c} \exp\left(-\frac{E_c}{kT_{mg}}\right) \quad (9)$$

The integral is evaluated according to the procedure suggested by Christodoulides [9] and 's' is evaluated using equation (9).

3. Results and discussions

A total number of 17 TL peaks corresponding to 4 phosphors which were reported in Ref. [2] have been analyzed in a systematic manner. The heating rate in all the cases is 1.0 K s^{-1} .

3.1. Gaussian curve fitting parameters

The TL curves (equation 1) generated by using GOK parameters of Ref. [2] have been fitted with the Gaussian function (3). Table 1 shows the input parameters (E, s, b) for all the peaks [2]) along with the parameters (I_0, A, σ, T_{mg}) of Gaussian functions (equation 3).

Table 1: The input values [2] for TL curve generation (equation 1) along with the Gaussian function parameters (equation 3). The linear heating rate is $\beta = 1.0\text{K.s}^{-1}$

Sample	Input parameters [2]				Present Gaussian curve parameters					
	Peak	E (eV)	s (s^{-1})	b	I_0	A	σ	T_{mg}	FOM (%)	χ^2
ZnS : Eu⁺, Co²⁺	P1	0.79	3.82e10	2.00	0.0108	44.5337	36.8755	340.94	7.43	3.36e-04
	P2	0.80	7.52e09	2.00	0.0155	49.8293	41.3912	365.63	6.95	3.98e-04
	P3	1.00	1.02e10	2.00	0.0169	59.8393	49.7542	448.38	6.70	4.08e-04
	P4	1.10	3.28e11	2.00	0.0151	52.307	43.4713	437.06	6.50	3.58e-04
Ca_{0.79}Sr_{0.21}S : Eu	P1	0.70	1.88e09	1.23	0.0052	38.4437	31.1458	336.43	9.59	5.68e-04
	P2	0.70	3.04e08	1.10	0.0063	41.5859	33.7484	361.15	12.19	1.08e-03
	P3	0.89	1.46e10	1.68	0.0115	48.3173	39.7807	394.61	4.41	1.38e-04
	P4	0.95	8.02e09	1.20	0.0071	45.1346	36.7273	427.63	10.32	8.06e-04
	P5	1.00	2.98e09	1.60	0.0150	58.2297	47.9792	467.50	3.62	1.35e-04
Sr₂MgSi₂O₇ : Eu²⁺Dy³⁺	P1	0.55	1.22e08	1.10	0.0040	36.0792	29.2260	297.79	12.26	8.10e-04
	P2	0.60	6.15e07	1.10	0.0054	41.2657	33.4532	333.38	11.91	9.68e-04
	P3	0.79	3.22e09	1.57	0.0091	46.6794	38.2830	371.42	4.48	1.39e-04
	P4	0.95	3.99e10	1.60	0.0098	46.9125	38.5215	405.51	4.50	1.44e-04
BaZrSi₃O₉ : Eu²⁺, Pr³⁺	P1	0.75	7.93e10	1.10	0.0041	29.7550	24.1389	314.01	13.52	8.63e-04
	P2	0.75	9.86e08	1.80	0.0145	51.5815	42.6531	370.47	4.85	2.09e-04
	P3	1.00	1.16e11	1.10	0.0060	37.8192	30.7238	409.47	13.32	1.11e-03
	P4	1.10	7.16e10	1.20	0.0073	44.5835	36.2954	456.60	10.66	8.45e-04

Table 1 also lists the FOM and χ^2 values. These FOM values are far higher than those reported in Ref. [2] where a different relation was used. We have checked that the expression used in Ref. [2] yields much higher FOM than the present values. The present values of FOM indicate that fitting of TL curves using Gaussian functions should not be treated as a good choice. It may be noted that only when b lies around 1.6 – 1.7, FOM is below 5% which also very high [10]. The χ^2 results are in excellent agreement with those obtained by Qti-Plot software. In Fig. 1, the TL curves and Gaussian fitting of Peak 3

of $\text{Ca}_{0.79}\text{Sr}_{0.21}\text{S} : \text{Eu}$ and Peak 1 of $\text{Sr}_2\text{MgSi}_2\text{O}_7 : \text{Eu}^{2+}\text{Dy}^{3+}$ are shown along with the respective residues.

The residue in Fig. 1(a) where $b = 1.68$ vary in a smaller range as compared to Fig. 1(b).

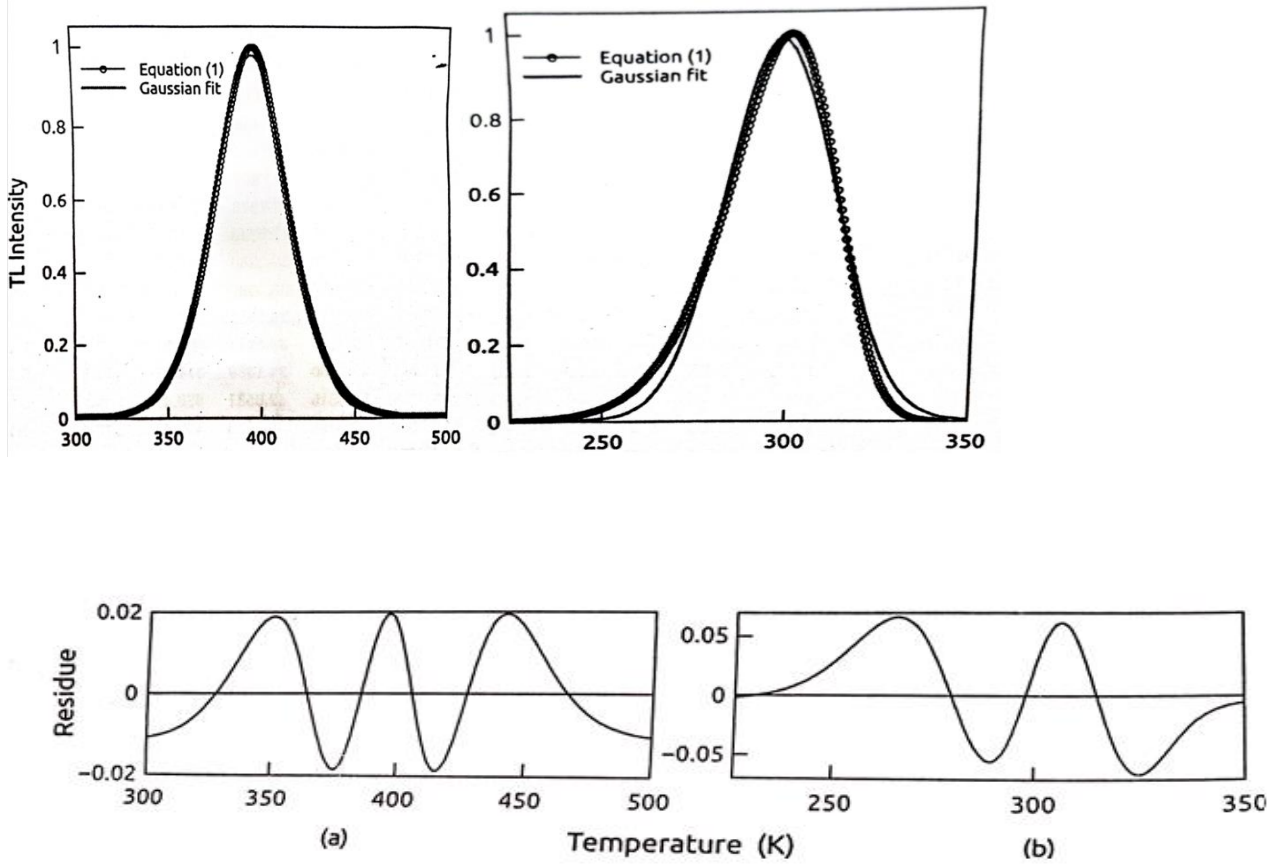


Figure 1: Fitting of TL glow curve using Gaussian functions and the residues. (a) Peak 3 of $\text{Ca}_{0.795}\text{Sr}_{0.215} : \text{Eu}$; input parameters: $E = 0.89 \text{ eV}$, $s = 1.46 \times 10^{10} \text{ s}^{-1}$ and $b = 1.684$; (b) Peak 1 of $\text{Sr}_2\text{MgSi}_2\text{O}_7 : \text{Eu}^{2+}, \text{Pr}^{3+}$; input parameters: $E = 0.55 \text{ eV}$, $s = 1.22 \times 10^8 \text{ s}^{-1}$ and $b = 1.10$.

In Table 2, we have given the results reported in Ref. [2] using Gaussian fitting and also listed the trapping parameters calculated in this work.

Table 2: Kinetic parameters derived by fitting with Gaussian function in the present work and comparison with the results reported in Ref. [2].

Sample	Peak	Gartia			Present					
		T_m (K)	E (eV)	s (s^{-1})	T_{mg} E _g (eV)	b	E _g (eV)	S_r (s^{-1})	S_c (s^{-1})	
ZnS : Eu²⁺, Co²⁺	P1	353.5	0.80	9.42e11	340.94	0.702	1.63	0.874	2.41e10	1.61e09
	P2	363.5	0.80	7.69e10	365.63	0.714	1.62	0.882	6.99e09	4.14e08
	P3	399.2	0.89	3.63e10	448.38	0.894	1.62	0.855	1.12e10	5.50e08
	P4	429.3	0.62	4.24e06	437.06	0.982	1.66	0.950	2.11e11	1.20e10
Ca_{0.75}Sr_{0.25}S : Eu²⁺	P1	337.4	0.70	3.04e10	336.43	0.823	1.68	0.802	2.13e12	1.72e11
	P2	368.3	0.88	8.39e10	361.15	0.874	1.68	0.850	1.57e12	1.17e11
	P3	402.8	1.37	1.46e16	394.61	0.877	1.66	0.847	1.56e11	9.78e09
	P4	431.4	1.66	2.66e18	427.63	1.131	1.71	1.109	2.16e13	1.49e12
	P5	467.0	1.60	1.23e16	467.50	1.015	1.65	0.980	8.93e10	4.61e09
Sr₂MgSi₂O₇ :										
Eu²⁺, Dy³⁺	P1	311.5	0.52	1.51e07	297.79	0.685	1.67	0.664	3.91e11	3.35e10
	P2	340.8	0.94	8.47e12	333.38	0.746	1.66	0.723	2.03e11	1.51e10
	P3	371.6	1.11	1.08e14	371.42	0.807	1.65	0.778	8.95e10	5.81e09
	P4	396.3	0.81	1.19e04	405.51	0.961	1.68	0.934	8.87e11	5.76e10
BaZrSi₃O₉ :										
Eu²⁺, Pr³⁺	P1	314.0	0.69	9.58e09	314.01	0.937	1.73	0.926	1.07e13	1.14e12
	P2	385.0	0.64	1.28e07	370.47	0.711	1.61	0.678	4.72e09	2.71e08
	P3	412.0	1.51	3.35e17	409.47	1.251	1.74	1.238	2.50e15	2.09e14
	P4	466.0	1.10	4.79e10	456.60	1.311	1.73	1.293	2.97e14	2.09e13

The peak positions (T_{mg}) listed in Table 2 show significant departures in many cases from T_m values of Ref. [2]. The estimation of other parameters are detailed below.

3.2. Determination of trapping parameters

Activation energy

Using $\mu_g = 0.5$ and the values of T_{mg} , $T_{\frac{1}{2}}^{\pm}$ (equation 6), the average activation energies (denoted as E_c) are estimated according to the peakshape relations [7] and are listed in Table 2. The E_c values are found to differ in many cases by a significant amount from those reported in Ref. [2], as is evident from Table 2. For example, in case of peak 4 of ZnS: Eu⁺, Co²⁺, Ref. [2] reports $T_m = 429.3$ K and $E = 0.62$ eV whereas the present calculations show that $T_{mg} = 437.06$ K and $E_c = 0.982$ eV. Similar departures in a random manner in other cases are evident from Table 2. The input values of activation energies do not agree quite well with the present Gaussian fitted results (E_c) which again reveals the limitation of Gaussian function for TL analysis.

Order of kinetics

The value of the order of kinetics(b) as determined by equation (7) is denoted by b' for all the peaks are listed in Table 2. Generally, a real TL curve is not symmetric and μ_g varies from 0.42 to 0.52 approximately as b' changes from 1 to 2. But a Gaussian curve is always symmetric. As a consequence, with $\mu_g = 0.5$, the b' values are not expected to differ much and this is evident from Table 2. The slight variation of b' is due to its weak dependence on $\frac{E_c}{kT_{mg}}$ and the present values are consistent with the observation of an earlier work of Gartia et al. [11]. The input values [2] of b' were within the range of 1.1 – 2.0 which are not reflected through the Gaussian fitted results.

These values of b' have been used to recalculate the activation energy using the expressions suggested by Gartia et al. [12] and the results (E_g) are listed in Table 2. These E_g values are found to be less than E_c by $\sim 5\%$ which is a consistent feature for all the peaks considered.

Frequency factor

The frequency factors estimated using equations (8) and (9) are given as s_r and s_c respectively in Table 2. We observe that the present values of s_r and s_c are found to differ by around one order of magnitude, s_r being consistently greater than s_c for all the peaks. However, in Ref. [2], the frequency factors derived from GOK method and Gaussian fitting differ in a random manner and in some cases, they differ by large orders of magnitude. It is to note that from theoretical point of view, equation (9) is more reliable for determination of frequency factors (s_c) while equation (8) is merely an empirical and

approximated approach. A closer look on Tables 1 and 2 reveals that the s_c values in some cases do not match at all with the input values of 's' used for peak generation. Peak no. 1, 2 and 4 of $\text{Ca}_{0.795}\text{Sr}_{0.215}\text{:Eu}$, peak 3 and 4 of $\text{Sr}_2\text{MgSi}_2\text{O}_7\text{:Eu}^{2+}\text{Dy}^{3+}$ and peak 1, 3 and 4 of $\text{BaZrSi}_3\text{O}_9\text{:Eu}^{2+}, \text{Pr}^{3+}$ show behaviour for which the 's' values lie within the range 1.10 – 1.23 [2], as is evident from Table 1. However, in all other cases, the ' s_c ' values (Table 2) is in fair agreement with the input values of 's' (Table 1) where 's' is close to 1.6 or even higher.

3.3. TL analysis using inflection points

The inflection points of Gaussian curve (equation 4) have been used to determine activation energy according to the methodology suggested by Gartia *et al.* [13] and Singh *et al.* [14]. The results (E_i) along with the relative intensities at $T = T_i^\pm$ are listed in Table 3.

Table 3: Results of glow curve analysis using inflection points of Gaussian curves.

Sample	Peak	T_{mg} (K)	T_i^- (K)	T_i^+ (K)	$I_i / I_m E_i$ (eV)	ΔE (%)
ZnS : Eu²⁺, Co²⁺P1						
P1	340.94	322.50	359.38	0.595	0.608	23.1
P2	365.63	344.93	386.32	0.598	0.618	22.8
P3	448.38	423.50	473.25	0.599	0.775	22.5
P4	437.06	415.32	458.79	0.598	0.858	22.0
Ca_{0.79} Sr_{0.21} S : Eu P1						
P1	336.43	320.85	352.00	0.602	0.717	2.4
P2	361.15	344.28	378.03	0.603	0.761	8.8
P3	394.61	374.72	414.50	0.599	0.763	14.3
P4	427.63	409.27	446.00	0.602	0.992	4.4
P5	467.50	443.51	491.49	0.602	0.885	11.5
Sr₂ MgSi₂ O₇ :						
Eu²⁺, Dy³⁺P1						
P1	297.79	283.18	312.40	0.601	0.594	8.0
P2	333.38	316.66	350.11	0.602	0.648	8.0
P3	371.42	352.28	390.56	0.599	0.700	11.4
P4	405.51	386.24	424.77	0.599	0.839	11.7
BaZrSi₃ O₉ :						
Eu²⁺, Pr³⁺P1						
P1	314.01	301.94	326.08	0.601	0.825	10.0
P2	370.47	349.15	391.80	0.600	0.614	18.2
P3	409.46	394.10	424.83	0.602	1.105	10.5
P4	456.60	438.46	474.75	0.602	1.156	5.1

For the experimental TL curves covered here, $\frac{T_i^-}{T_i^+}$ ratio is expected to lie within the range 0.53 – 0.69 (increasing) in the falling side and 0.69 – 0.64 (decreasing) in the rising side as ‘ s_c ’ increases from 1.1 to 2.0 [13, 15], with a margin of error around 5% depending on the value of $\frac{E}{kT_m}$. However, the Gaussian curves being symmetric, the intensity (I_i) at both the inflection points ($T_i^\pm$) are same and the ratio $\frac{T_i^-}{T_i^+}$ for all the peaks in this case is around 0.6 (Table 3). The E_i values do not show any acceptable agreement with the input values. In Table 1, we have given the error

$$\Delta E_i = \frac{|E - E_i|}{E} \times 100\% \quad (10)$$

for these E_i values. Tables 1 and 3 reveal that for ‘ s ’ lying within 1.1 – 1.23, the E_i values have less departure from the input values. It is remarkable that the frequency factors showed gross departure for this range of ‘ s ’, as mentioned before. For other cases, E_i deviates from the input value by a large amount, sometimes even more than 20%. Such results are hence inconclusive for determining the trapping parameters from Gaussian curve fitting.

4. Concluding remarks

The present work reviews the applicability of Gaussian functions for fitting TL curves. It can be concluded from many aspects that Gaussian functions are not a suitable choice for fitting TL curves. The FOM values become very large in such cases. The fundamental problem lies with the symmetry of the curve. A TL curve in general is not symmetric and has asymmetry has a strong dependence on the order of kinetics. This feature is not reflected when fitted with a Gaussian function which is ideally symmetric. The parameters are derived by using standard TL analysis techniques do not show consistent agreement with the input parameters used by Chandrasekhar and Gartia [2] and it appears that some inadvertent errors might have been crept in their calculations.

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Fractional Dual-Phase-Lag Thermoelasticity in Spherical Shells: A New Perspective

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Abstract

This research introduces a fractional dual-phase-lag (FDPL) thermoelastic model to investigate the behavior of a spherical shell. The model integrates the two-temperature theory to address the difference between conductive and thermodynamic temperatures while utilizing non-singular kernels to provide realistic and stable responses to transient phenomena. Incorporating fractional calculus into the dual-phase-lag framework enables the model to capture memory-dependent effects, offering a deeper insight into heat conduction and stress-wave interactions in thermoelastic materials. The Laplace transform technique is employed to derive analytical solutions. A comprehensive analysis is conducted on the impact of fractional-order parameters, two-temperature effects, and kernel selections on the thermoelastic response.

Key words: Two-temperature Generalized thermo-elasticity; fractional derivative; dual-phase lag (DPL) model; State-Space Approach; Numerical inversion of the Laplace transform.
MSC74F, MSC31B10, MSC31B25

1. Introduction

Classical calculus defines derivatives of integer order to describe the rate of change of a function. Fractional calculus generalizes this using non-singular kernels to enable differentiation of non-integer orders. Such kernels—often involving functions like gamma, exponential, or Mittag-Leffler—capture memory and non-local effects, making them valuable in modeling complex systems in hydrodynamics, viscoelasticity, physics, biology, and engineering. The development of new fractional derivatives with well-defined mathematical properties and practical relevance remains a key research focus.

The Riemann–Liouville fractional integral and its derivative of order $\alpha > 0$ of a function $f(t)$, starting from a point a , are

$${}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau$$

$${}_a D_t^\alpha f(t) = \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau \right]$$

The Caputo fractional derivative [1] is

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} f(\tau) d\tau$$

The Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ is defined as:

$${}_a^H D_t^{\alpha,\beta} f(t) = {}_a I_t^{\beta(1-\alpha)} \left(\left(\frac{d}{dt} \right)^n \left[{}_a I_t^{(1-\beta)(1-\alpha)} f(t) \right] \right)$$

The Caputo–Fabrizio fractional derivative [2] is a non-singular kernel fractional derivative that uses an exponential kernel instead of a power-law kernel, which avoids singularities at $t = \tau$.

For a function $f(t)$, the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$ is defined as :

$${}^{CF} D_t^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \frac{d}{d\tau} f(\tau) d\tau$$

The Atangana–Baleanu fractional time derivative, introduced in 2016, is a relatively recent concept in the field of fractional calculus. For a function $f(t)$, the Atangana–Baleanu fractional derivative of order $\alpha \in (0,1)$ is defined as:

$${}^{AB} D_t^\alpha f(t) = \frac{1}{1-\alpha} \int_0^t E_\alpha \left(-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) \frac{d}{d\tau} f(\tau) d\tau$$

The function $E_\alpha(z)$ denotes One-parameter Mittag-Leffler function and it is expressed as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

where $z \in \mathbb{C}$ and $0 < \alpha < 1$. Many researchers have studied these applications in recent years [3-10].

Gurtin and Williams [11,12] argued that the second law of thermodynamics for continuous media should not be restricted to a single-temperature formulation. To address this, [13-15] developed the two-temperature theory (2TT), introducing the conductive temperature ϕ and thermodynamic temperature T . The discrepancy between them is characterized by a non-negative parameter α ; when $\alpha = 0$, the theory reduces to classical thermoelasticity (CTE). While $\phi = T$ in the absence of heat supply under steady-state conditions, they generally differ in transient processes and wave propagation. Despite its broader framework, both CTE and 2TT retain the classical paradox of predicting infinite thermal disturbance propagation speeds [16].

Tzou introduced the dual-phase-lag (DPL) model by incorporating distinct phase lags for the heat flux and temperature gradient. Subsequent studies established the existence, uniqueness, reciprocity, and spatial behavior of solutions within the two-temperature theory (2TT) framework [17-23]. Recent advances have extended thermoelasticity through fractional-order operators, enabling a more accurate description of memory-dependent heat transfer and thermo-mechanical coupling in complex materials.

In this work, a fractional two-temperature dual-phase-lag thermoelastic model is employed to investigate thermal shock responses in a spherically isotropic steel shell. The governing equations are solved in the Laplace domain using the state-space approach [24], with numerical inversion performed via the Fourier-series method [25].

2. Basic equations and constitutive relations

A spherically symmetric shell with inner and outer radii a and b , respectively, is initially at a uniform temperature T_0 and subjected to thermal shock at its inner surface. Owing to spherical symmetry, the displacement and temperature fields depend solely on the radial coordinate r and time t .

The stress-strain-temperature relations in the present problem are

$$\sigma_{rr} = C_{11} \frac{\partial u}{\partial r} + 2C_{12} \frac{u}{r} - \beta_r T, \quad (1)$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = C_{12} \frac{\partial u}{\partial r} + (C_{22} + C_{23}) \frac{u}{r} - \beta_\theta T, \quad (2)$$

and generalized heat conduction equation is

$$\left(1 + \tau_q^\alpha D_t^{(\alpha)}\right) \left[\rho C_e \ddot{T} + T_0 \frac{\partial^2}{\partial t^2} \left(\beta_r \frac{\partial u}{\partial r} + \beta_\theta \frac{2u}{r}\right)\right] = K \left(1 + \tau_\phi^\alpha D_t^{(\alpha)}\right) \nabla^2 \phi. \quad (3)$$

Here, σ_{ij} ($i, j = r, \theta$) represents the components of the stress tensor, T is the thermodynamic temperature relative to the reference temperature T_0 , and ϕ denotes the conductive temperature. The constants C_{11} , C_{12} , C_{22} , and C_{23} are elastic moduli specific to steel. β_r and β_θ are thermal moduli, K is the radial thermal conductivity. ρ is the mass density, and C_e is the specific heat at constant strain. The parameters τ_ϕ and τ_q represent the phase-lags of the temperature gradient and heat flux, respectively.

The framework encompasses the classical thermoelasticity (CTE) model ($\tau_\phi = \tau_q = 0$), the Lord–Shulman (LS) formulation incorporating a single thermal delay ($\tau_\phi = 0$), and the dual-phase lag (DPL) model accounting for both τ_ϕ and τ_q .

The equation of motion for stress in spherical polar coordinates is expressed as follows:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = \rho \frac{\partial^2 u}{\partial t^2}. \quad (4)$$

The relationship between the thermodynamic temperature T and the conductive temperature ϕ is given by

$$\phi - T = \alpha_1 \nabla^2 \phi, \quad (5)$$

where α_1 is the temperature discrepancy and ∇^2 is the Laplacian, given by

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right).$$

For steel $\beta_r = \beta_\theta$ [23].

By eliminating T using Equations (3) and (5), we obtain:

$$K \left(1 + \tau_\phi^\alpha D_t^{(\alpha)}\right) \frac{\partial}{\partial t} \nabla^2 \phi + \alpha_1 \rho C_e \left(1 + \tau_q^\alpha D_t^{(\alpha)}\right) \nabla^2 \phi =$$

$$(1 + \tau_q^\alpha D_t^{(\alpha)}) \frac{\partial^2}{\partial t^2} \left[\rho C_e \phi + \beta_r T_0 \left(\frac{\partial u}{\partial r} + \frac{2u}{r} \right) \right]. \quad (6)$$

using relations (1) and (2) in Equation (4) we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2(C_{22} + C_{23} - C_{12})}{C_{11}} \frac{u}{r^2} - \frac{\beta_r}{C_{11}} \frac{\partial}{\partial r} \left[\phi - \alpha_1 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right) \right] = \frac{\rho}{C_{11}} \frac{\partial^2 u}{\partial t^2}. \quad (7)$$

By introducing the following non-dimensional variables:

$$r' = c_0 \eta r, \quad u' = c_0 \eta u, \quad t' = c_0^2 \eta t, \quad \tau'_T = c_0^2 \eta \tau_T, \quad \tau'_q = c_0^2 \eta \tau_q, \quad \tau'_v = c_0^2 \eta \tau_v, \quad T' = \frac{T}{T_0},$$

$$\phi' = \frac{\phi}{T_0}, \quad \sigma'_{rr} = \frac{\sigma_{rr}}{C_{11}}, \quad \sigma'_{\theta\theta} = \frac{\sigma_{\theta\theta}}{C_{11}}, \quad a' = c_0 \eta a, \quad b' = c_0 \eta b, \quad K'^* = \frac{K^*}{K c_0^2 \eta},$$

where

$$c_0^2 = \frac{C_{11}}{\rho} \quad \text{and} \quad \eta = \frac{\rho C_e}{K}$$

and omitting the primes, relations (1), (2) and Equations (7), (6) and (5) can be written, respectively, in the following forms :

$$\sigma_{rr} = \frac{\partial u}{\partial r} + C_2 \frac{u}{r} - a_1 T, \quad (8)$$

$$\sigma_{\theta\theta} = C_2 \frac{\partial u}{\partial r} + \frac{C_{22} + C_{23}}{C_{11}} \frac{u}{r} - a_1 T, \quad (9)$$

$$\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} - \frac{2(C_{22} + C_{23} - C_{12})}{C_{11}} \frac{u}{r^2} \right) - a_1 \left[\frac{\partial \phi}{\partial r} - \omega \frac{\partial}{\partial r} \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right) \right] = \frac{\partial^2 u}{\partial t^2}, \quad (10)$$

$$\left[\left(1 + \tau_q^\alpha D_t^{(\alpha)} \right) \frac{\partial}{\partial t} + \omega \frac{\partial^2}{\partial t^2} \left(1 + \tau_q^\alpha D_t^{(\alpha)} \right) \right] \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right) = \frac{\partial^2}{\partial t^2} \left(1 + \tau_q^\alpha D_t^{(\alpha)} \right) \left[\phi + a_2 \left(\frac{\partial u}{\partial r} + \frac{2u}{r} \right) \right], \quad (11)$$

and

$$\phi - T = \omega \nabla^2 \phi, \quad (12)$$

where $C_2 = \frac{C_{12}}{C_{11}}$, $a_1 = \frac{\beta_r T_0}{C_{11}}$, $\omega = \alpha_1 c_0^2 \eta^2$, $a_2 = \frac{\beta_r}{K \eta}$.

Since the medium is initially quiescent and undisturbed, all state variables are taken to be zero at the initial moment.

The mechanical and thermal boundary conditions are defined as follows:

$$e = 0, 0.1 \quad \text{on} \quad r = a, b; \quad (13)$$

$$\begin{aligned} \phi &= \chi_1 H(t), \quad \text{on} \quad r = a; \\ \phi &= 0, \quad \text{on} \quad r = b. \end{aligned} \quad (14)$$

where χ_1 is a dimensionless constant and $H(t)$ is the Heaviside's unit step function.

3. Method of Solution

To solve the governing equations, the Laplace transform with respect to time t is defined as:

$$\{\bar{u}(r, p), \bar{\phi}(r, p)\} = \int_0^\infty \{u(r, t), \phi(r, t)\} e^{-pt} dt, \quad \text{Re}(p) > 0 \quad (15)$$

where p is the transform parameter. Applying Eq. (15) to the constitutive and governing equations yields the transformed radial and hoop stresses:

$$\bar{\sigma}_{rr} = \frac{d\bar{u}}{dr} + 2C_2 \frac{\bar{u}}{r} - a_1 \bar{T}, \quad (16)$$

$$\bar{\sigma}_{\theta\theta} = C_2 \frac{d\bar{u}}{dr} + \frac{C_{22}+C_{23}}{C_{11}} \frac{\bar{u}}{r} - a_1 \bar{T}, \quad (17)$$

along with the transformed temperature field relation $\bar{T} = (1 - \omega \nabla^2) \bar{\phi}$. Here, $\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}$ denotes the spherical Laplacian operator.

By introducing the volume dilatation $\bar{e} = \frac{d\bar{u}}{dr} + \frac{2\bar{u}}{r}$ and invoking the material relation for steel ($C_{22} + C_{23} = C_{11} + C_{12}$), the coupled field equations simplify into a set of second-order differential equations:

$$\nabla^2 \bar{\phi} = C \bar{\phi} + a_2 C \bar{e}, \quad (18)$$

$$\nabla^2 \bar{e} = M_1 \bar{\phi} + M_2 \bar{e}, \quad (19)$$

where the parameters are defined compactly as:

$$C = \frac{p^2 l_q}{l_{\phi p} + \omega p^2 l_q}, \quad M_1 = Cl, \quad M_2 = a_2 Cl + \kappa^2 \quad (20)$$

with

$$l_{\phi,q} = 1 + \tau_{\phi,q} \alpha \frac{p^\alpha}{\alpha + p^\alpha(1-\alpha)}, \quad l = \frac{a_1(1-C\omega)}{1+C\omega a_1 a_2}, \quad \kappa^2 = \frac{p^2}{1+C\omega a_1 a_2} \quad (21)$$

The transformed boundary conditions on $r = a, b$ are written as:

$$\bar{e} = \left\{0, \frac{0.1}{p}\right\}, \quad \bar{\phi} = \left\{\frac{\chi_1}{p}, 0\right\} \quad (22)$$

Equations (18) and (19) can be compactly organized into the vector matrix differential equation:

$$\nabla^2 \bar{V}(r, p) = A(p) \bar{V}(r, p) \quad (23)$$

where the state vector $\bar{V}(r, p)$ and the system matrix $A(p)$ are given by:

$$\bar{V}(r, p) = \begin{bmatrix} \bar{\phi}(r, p) \\ \bar{e}(r, p) \end{bmatrix}, \quad A(p) = \begin{bmatrix} C & a_2 C \\ M_1 & M_2 \end{bmatrix} \quad (24)$$

4. State-space Approach

The general solution to the matrix differential equation (23) can be formulated using matrix exponentials. Applying the boundary conditions at the inner and outer surfaces, $\bar{V}(a, p)$ and $\bar{V}(b, p)$, the solution takes the form of matrix hyperbolic functions:

$$\bar{V}(r, p) = \frac{a \sinh[\sqrt{A(p)}(b-r)] \bar{V}(a, p) + b \sinh[\sqrt{A(p)}(r-a)] \bar{V}(b, p)}{r \sinh[\sqrt{A(p)}(b-a)]}. \quad (25)$$

To evaluate the matrix functions analytically, we determine the eigenvalues of $A(p)$. The characteristic equation of $A(p)$ is given by:

$$\lambda^4 - \lambda^2(C + M_2) + (CM_2 - a_2 CM_1) = 0, \quad (26)$$

whose roots, λ_1^2 and λ_2^2 , yield the eigenvalues λ_1 and λ_2 for the matrix $\sqrt{A(p)}$.

Invoking the Cayley-Hamilton theorem, any analytical matrix function $F(\sqrt{A(p)})$, such as the hyperbolic sines in Eq. (25), can be expressed as a linear polynomial of $\sqrt{A(p)}$:

$$F(\sqrt{A(p)}) = \alpha_{10}I + \alpha_{11}\sqrt{A(p)}, \quad (27)$$

where the scalar coefficients α_{10} and α_{11} are obtained by substituting the eigenvalues λ_1 and λ_2 into Eq. (26).

By expanding the terms $\sinh[\sqrt{A(p)}(b-r)]$, $\sinh[\sqrt{A(p)}(r-a)]$, and the inverse matrix $\{\sinh[\sqrt{A(p)}(b-a)]\}^{-1}$ using this theorem, the state vector can be decoupled into the compact transfer-matrix form:

$$\begin{bmatrix} \bar{\phi}(r, p) \\ \bar{e}(r, p) \end{bmatrix} = \frac{a}{r} Q(r, p) \begin{bmatrix} \bar{\phi}(a, p) \\ \bar{e}(a, p) \end{bmatrix} + \frac{b}{r} R(r, p) \begin{bmatrix} \bar{\phi}(b, p) \\ \bar{e}(b, p) \end{bmatrix}, \quad (28)$$

where $Q(r, p)$ and $R(r, p)$ are 2×2 matrices whose algebraic elements are analytically determined from the characteristic roots λ_1 and λ_2 and the boundary geometry. Eq. (28) yields the explicit solutions for the physical fields $\bar{\phi}$ and $\bar{e}$.

Subsequently, utilizing Eq. (18) and $\bar{T} = (1 - \omega \nabla^2) \bar{\phi}$, the thermodynamic temperature in the Laplace domain is obtained as:

$$\bar{T} = (1 - C\omega) \bar{\phi} - a_2 C \omega \bar{e}. \quad (29)$$

Finally, substituting $\bar{e}$, $\bar{u}$, and $\bar{T}$ into the constitutive relations, the radial and hoop stress components are determined:

$$\bar{\sigma}_{rr} = \bar{e} + (C_2 - 1) \frac{2\bar{u}}{r} - a_1 \bar{T}, \quad (30)$$

$$\bar{\sigma}_{\theta\theta} = C_2 \bar{e} + \left(\frac{C_{22} + C_{23}}{C_{11}} - 2C_2 \right) \frac{\bar{u}}{r} - a_1 \bar{T}. \quad (31)$$

This completes the closed-form solution in the Laplace transform domain.

5. Numerical Inversion of the Laplace Transform

Due to the complexity of the analytical solutions in the Laplace domain, a numerical inversion technique based on a Fourier series expansion is employed to transform the field variables back to the time domain. The time-dependent function $f(r, t)$ is approximated within the interval $[0, 2T]$ by the truncated series:

$$f_N(r, t) = \frac{e^{\gamma t}}{T} \left[\frac{1}{2} \bar{f}(r, \gamma) + \text{Re} \sum_{k=1}^N \bar{f} \left(r, \gamma + \frac{ik\pi}{T} \right) e^{\frac{ik\pi t}{T}} \right], \quad (32)$$

where γ is a scaling parameter chosen to control the discretization error, and $p = \gamma + \frac{ik\pi}{T}$ represents the discrete complex Laplace variables.

To systematically reduce the discretization error, the 'Korrektur' method is implemented:

$$f_{NK}(r, t) = f_N(r, t) - e^{-2\gamma T} f_{N'}(r, 2T + t), \quad (33)$$

where N' is an integer tracking a subset of terms ($N' < N$).

The convergence of the series in Eq. (32) is accelerated using the non-linear ε -algorithm. Let s_m denote the m -th partial sum of the series. The sequence elements are iteratively computed using the following routing rule:

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + \frac{1}{\varepsilon_{p,m+1} - \varepsilon_{p,m}}, \quad (34)$$

with the initializations $\varepsilon_{0,m} = 0$ and $\varepsilon_{1,m} = s_m$. The sequence step $\varepsilon_{N,1}$ yields the final accelerated numerical approximation for $f(r, t)$.

6. Numerical results and discussions

The temperature, stress, and strain distributions in the space-time domain are computed numerically using a Fourier series expansion technique [25]. A custom numerical code, developed in Fortran 90, is used for these calculations. The numerical calculations are performed for steel with material constants adopted from Islam et al. [23]:

$$\begin{aligned} C_{11} &= 0.27865 \times 10^{12}, & C_{12} &= 0.11942 \times 10^{12}, & \rho &= 0.78 \times 10^4, \\ C_e &= 117, & \beta_r &= 5.5889 \times 10^6, & K &= 59, \\ T_0 &= 296, & \omega &= 0.1, & \chi_1 &= 3, \end{aligned}$$

where the units are SI. The dimensionless inner and outer radii of the spherical shell are taken as ($a = 1$) and ($b = 3$), respectively.

Figure 1 illustrates the variation of the dimensionless radial stress σ_{rr} with the radial coordinate r for different fractional orders α under the Lord–Shulman (LS) and Dual-Phase-Lag (DPL) models. It is observed that the radial stress attains its maximum magnitude near the heated inner boundary ($r = 1$) due to the imposed thermal shock. As the radial distance increases, the stress amplitude decreases rapidly and approaches zero towards the outer boundary. The stress field exhibits wave-like behavior with alternating compressive and tensile regions, reflecting the thermoelastic wave propagation inside the spherical shell. The fractional parameter α has a pronounced influence on the stress response. For both LS and DPL models, larger values of α produce higher stress amplitudes, whereas decreasing α reduces the peak stress significantly. This behavior indicates that stronger memory effects (smaller α) suppress thermoelastic wave intensity and enhance thermal damping.

A comparison between the two models reveals that the DPL model predicts lower stress magnitudes and smoother stress profiles than the LS model. The introduction of the additional phase lag associated with the temperature gradient in the DPL theory increases the thermal relaxation effect and attenuates stress oscillations. Consequently, the DPL model provides a more realistic description of finite-speed heat transport and thermoelastic wave propagation.

Figure 2 depicts the variation of the conductive temperature ϕ with respect to the radial coordinate r for various fractional orders α under the LS and DPL theories. The conductive temperature is maximum at the inner surface where the thermal shock is applied and decreases gradually towards the outer surface. A thermal wave front can be clearly identified, beyond which the temperature approaches zero, indicating finite thermal disturbance propagation. The influence of the fractional parameter is significant. Higher values of α lead to larger temperature amplitudes and stronger

thermal waves, whereas smaller values of α reduce the conductive temperature throughout the shell. This reduction arises from the enhanced memory effect associated with lower fractional

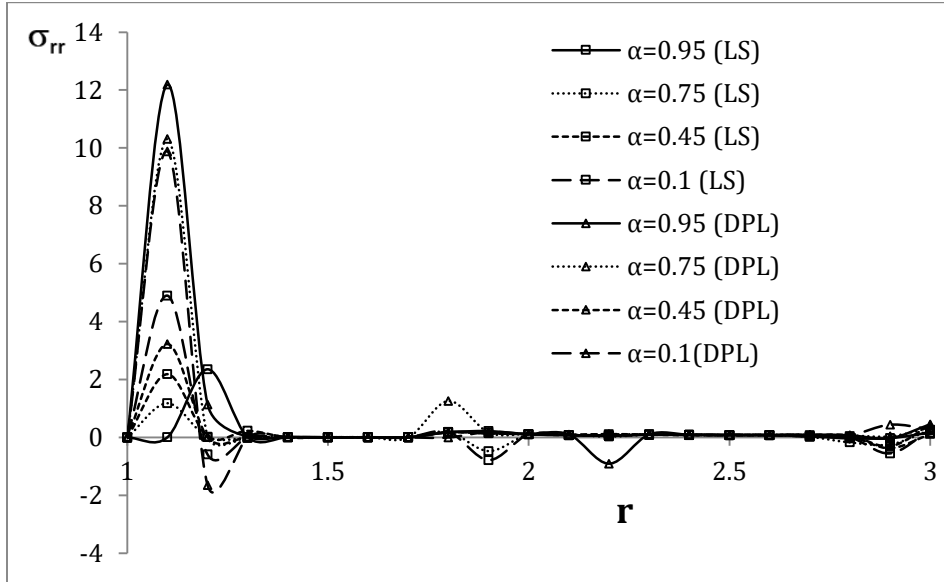


Fig. 1 Variation of radial stress σ_{rr} against radial distance r for different α with LS and DPL models ($t=1.2$)

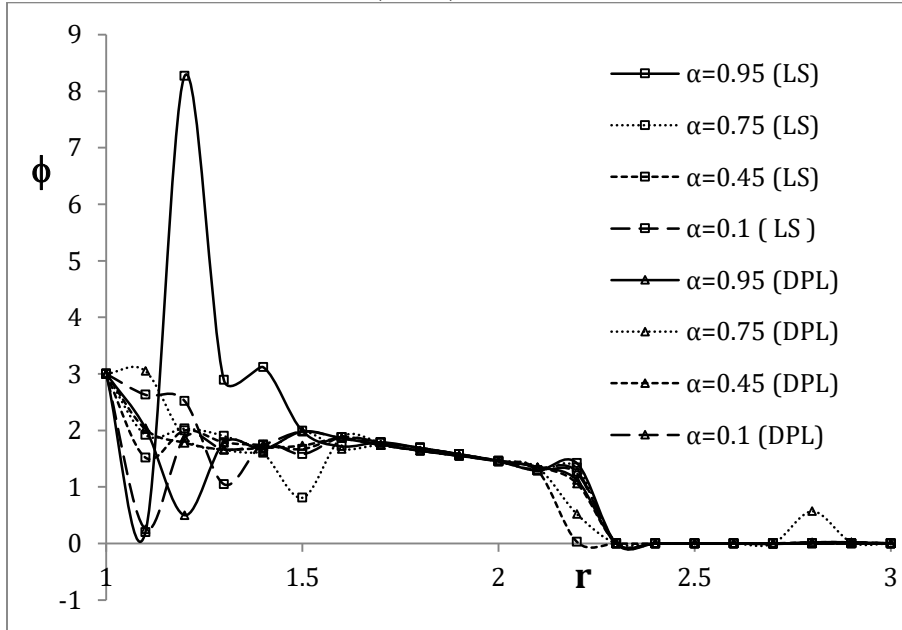


Fig. 2 Variation of conductive temperature ϕ against radial distance r for different α with LS and DPL models ($t=1.2$)

orders, which slows the thermal diffusion process. Comparing the LS and DPL results, the DPL model exhibits smoother temperature distributions and lower oscillatory behavior than the LS model. The additional phase lag incorporated in the DPL model delays heat conduction and

increases energy dissipation, thereby reducing the thermal wave amplitude. Hence, the DPL theory predicts a more attenuated and physically realistic thermal response.

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Investigation of the effect of wall-slip on pulsatile Herschel Bulkey blood flow through a narrow stenosed arterial segment in presence of Body acceleration

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Abstract

This study investigates the influence of wall slip on pulsatile blood flow through a narrow stenosed artery in presence of periodic body acceleration. Blood is modelled as a Herschel-Bulkley fluid to accounts for its non-Newtonian behaviour, particularly at low shear rates. The model incorporates wall slip at the arterial boundary, which is significant in narrow arteries and under pathological conditions. The governing equations are formulated by considering pulsatile pressure gradient together with external body acceleration and are solved subject to appropriate boundary conditions. The effects of wall slip, yield stress, flow behaviour index and body acceleration on the velocity distribution and volumetric flow rate are investigated. The results demonstrate that wall slip significantly influences the flow characteristics, leading to enhanced velocity and altered flow patterns within the stenosed artery. Furthermore, increasing wall slip delays the onset of the flow reversal during the pulsatile cycle, indicating its important role in modulating the hemodynamic behaviour. The Herschel-Bulkley model provides a more general and realistic description of blood rheology than simpler non-Newtonian models, making it suitable for studying blood flow in narrow arteries.

Key words: wall slip, body acceleration, Herschel- Bulkley fluid, flow behaviour index, stenosed artery.

1.Introduction:

In most cardiovascular fluid dynamics studies, the no-slip boundary condition remains valid throughout much of the arterial system. In large and medium - sized arteries, wall slip is generally negligible. However, slip effects become important in small arteries (Internal diameter approximately 100 μm to 1mm), arterioles and capillaries, where micro scale phenomena influence blood flow. Slip is also significant when blood flows through porous or specially coated surfaces, and it plays an important role in artificial blood vessels, stents and other biomedical devices.

Wall slip effects become even more relevant when blood is modelled as a non-Newtonian fluid. Blood exhibits anomalous viscous properties because it is a suspension of cells in plasma, while plasma behaves as Newtonian fluid while blood often displays non-Newtonian properties, especially at low shear rates, as observed in small arteries and downstream of stenosed regions.

Several researchers have incorporated wall-slip condition into mathematical model of blood flow. Mishra and Shit [1] developed a mathematical model for blood flow through a stenosed arterial segment by considering wall slip and treating blood as a non-Newtonian fluid in the constricted region Panalagusamy [2] , Biswas and Chakraborty [3] , Sanyal [4] also investigated blood flow through stenosed arteries with wall-slip condition considering blood as Newtonian fluid.

The cyclic nature of heart pumping leads to pulsatile conditions in blood flow. Rabby et.al [5] suggested that during pulsatile motion, blood flow tends to have a low Reynolds number, particularly during diastole, early systole and within stenosed regions. Under these conditions the non-Newtonian behaviour of blood becomes especially significant. Chien [6] observed that the non –Newtonian properties of blood are more pronounced under pathological conditions such as Cerebro vascular disease, myocardial infraction and hypertension. Consequently, many researchers have investigated pulsatile blood flow stenosed arteries by modelling blood as non-Newtonian fluid .

Among the various non-Newtonian models the Casson and Herschel-Bulkley fluid are widely used. Nagarani and Sorojamma[7] studied blood flow through stenosed narrow arteries by considering blood as Casson fluid .D.S.Sankar and Lee [8] modelled Pulsatile flow of Non-Newtonian fluid in stenosed arteries Tu and Deville [9] showed that , at moderate shear rates, blood follows the Casson model .Chaturani and Ponnellagar Samy [10] observed that, in narrow diameter arteries , the Herschel-Bulkley model (H-B model)describes blood flow more accurately than the power –law and Bingham models. Lida [11] explained that ,in arterioles (diameter less than 0.1 mm)blood exhibits Casson fluid characteristics, whereas in larger arterioles and small arteries, the H-B model provides a better representation of blood rheology.

Since the H-B fluid models more general than Casson fluid model, it is more suitable for describing blood flow through narrow arteries .

In everyday life, the human body is often subjected to periodic acceleration during various activities such as sports, high-speed travel and flying. Arntzenius et al [12] observed that external disturbances acting on the human body have a significant effect on blood circulation. Sud and Sekhan [13] developed a mathematical model for blood flow in arteries under a pulsatile pressure gradient

and periodic acceleration. Chaturani and Palanisamy [14] investigated pulsatile blood flow through rigid tubes by considering body acceleration and later incorporated non-Newtonian characteristics into their model [15].

D.S.Sankar and A.P.M Ismail [16] developed a mathematical model for pulsatile blood flow through a stenosed artery by modelling blood as a H-B fluid at low shear rates, in the presence of body acceleration.

In the present study, we investigate the effect of wall slip on pulsatile blood flow through a narrow artery in the presence of body acceleration, considering blood as H-B fluid.

2. Mathematical formulation:

Consider an axially symmetric incompressible laminar pulsatile and fully developed flow of blood through a narrow arterial segment with stenosis. The geometry of the arterial segment is shown in fig 1. Assume that the blood is non-Newtonian and modelled as H-B fluid. The segment of a small artery is considered where wall of the stenosed artery has slip velocity. The governing equations are

$$\bar{\rho} \frac{\partial \bar{u}}{\partial \bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}) + \bar{F}(\bar{t}) \quad (2.1)$$

$$\frac{\partial \bar{p}}{\partial \bar{r}} = 0 \quad (2.2)$$

Where at any time $\bar{t}$, the axial component of velocity is $\bar{u}$, pressure is $\bar{p}$, the body acceleration is $\bar{F}(\bar{t})$ and $\bar{\tau} = |\bar{\tau}_{\bar{r}\bar{z}}| = -\bar{\tau}_{\bar{r}\bar{z}}$ is the shear stress. The fundamental equation for H-B fluid is given by

$$\begin{aligned} \bar{\tau} &= \bar{\mu}_H^{1/n} \left(-\frac{\partial \bar{u}}{\partial \bar{r}} \right)^{1/n} + \bar{\tau}_H \text{ for } \bar{\tau} > \bar{\tau}_H \\ &= \bar{\tau}_H \text{ for } \bar{\tau} \leq \bar{\tau}_H \end{aligned} \quad (2.3)$$

Where $\bar{\tau}_H, \bar{\mu}_H$ represents yield stress and coefficient of viscosity respectively for H-B fluid.

The geometry of the stenosed arterial segment can be formulated by

$$\begin{aligned} \bar{R}(\bar{z}) &= \bar{R}_0 - \bar{\delta} \left[1 + \cos \left(\frac{\pi \bar{z}}{2\bar{z}_0} \right) \right] \text{ if } |\bar{z}| \leq 2\bar{z}_0 \\ &= \bar{R}_0 \text{ otherwise} \end{aligned} \quad (2.4)$$

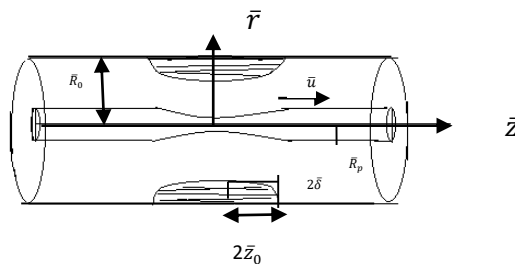


Figure 1

Figure 1 describes the geometry of the stenosed arterial segment ,Where $\bar{\delta}$ is the half of the maximum projection of the stenosis starting from the wall of the arterial segment to the lumen of the artery such that $\bar{\delta}/\bar{R}_0 \ll 1$, $\bar{R}(\bar{z})$ denotes the radius of the artery in the stenosed region. $\bar{R}_0$ is the radius of the normal artery. The boundary condition of the flow is given by

$$\begin{aligned} \text{i)} \quad & \bar{\tau} \text{ is finite at } \bar{r} = 0 \\ \text{ii)} \quad & \bar{u} = \bar{u}_s \quad \text{at } \bar{r} = \bar{R}(\bar{z}), \\ \text{iii)} \quad & \frac{\partial \bar{u}}{\partial \bar{r}} = 0 \quad \text{at } \bar{r} = 0, \end{aligned} \quad (2.5)$$

where, $\bar{u}_s$ is the slip velocity at the stenotic wall.

Pressure gradient can be defined as

$$-\frac{\partial \bar{p}}{\partial \bar{z}} = A_0 + A_1 \cos(\bar{\omega}_p \bar{t}) \quad (2.6)$$

The body acceleration in the axial direction is given by

$$F(t) = \bar{a}_0 \cos(\bar{\omega}_p \bar{t} + \varphi) \quad (2.7)$$

Where $\bar{\omega}_p = 2\pi \bar{f}_b$, $\bar{a}_0$ is the amplitude, φ is the lead angle and $\bar{f}_b$ is the frequency of the body acceleration. To obtain the equations, body acceleration and boundary conditions in dimensionless form we introduce non-dimensional variables

$$\begin{aligned} z = \frac{\bar{z}}{\bar{R}_0}, \tau = \frac{\bar{\tau}}{\frac{\bar{R}_0 \bar{A}_0}{4}}, \alpha^2 = \frac{\bar{R}_0^{n+1} \bar{\omega}_p \bar{\rho} \bar{A}_0^{n-1}}{2(2n-1)\bar{\mu}}, B = \frac{A_0}{A_0}, e = \frac{A_p}{A_0}, t = \bar{t} \bar{\omega}_p, u = \frac{2^{2n} \bar{\mu} \bar{u}}{A_0^n \bar{R}_0^{n+1}}, \\ \tau_H = \frac{\bar{\tau}_H}{\frac{\bar{R}_0 \bar{A}_0}{4}}, r = \frac{\bar{r}}{\bar{R}_0}, \omega = \frac{\bar{\omega}_b}{\bar{\omega}_p}, R(z) = \frac{\bar{R}(\bar{z})}{\bar{R}_0}, \delta s = \frac{\bar{\delta} s}{\bar{R}_0}, \end{aligned} \quad (2.8)$$

where α is the pulsatile Reynolds number or generalized Womersley frequency parameter. Using (2.8) the governing equation (2.1), (2.2), fundamental equation for H-B fluid (2.3), body acceleration (2.7) takes the following non-dimensional forms

$$\alpha^2 \frac{\partial u}{\partial t} = 2(1 + e \cos t) - \frac{1}{2r} \frac{\partial}{\partial r} \left[r \left(-\frac{\partial u}{\partial r} \right)^{\frac{1}{n}} + r \tau_H \right] + 2B \cos(\omega t + \varphi) \quad (2.9)$$

$$\frac{\partial p}{\partial r} = 0 \quad (2.10)$$

$$\begin{aligned} \tau &= \left(-\frac{\partial u}{\partial r} \right)^{\frac{1}{n}} + \tau_H \quad \text{for } \tau > \tau_H \\ &= \tau_H \quad \text{for } \tau \leq \tau_H \end{aligned} \quad (2.11)$$

$$F(t) = A_0 B \cos(\omega t + \varphi) \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\alpha^2}{2\bar{\rho}} \frac{\partial u}{\partial t} \quad (2.12)$$

The boundary condition (2.5) in non-dimensional form becomes

$$\begin{aligned} \text{i)} \quad & \tau \text{ is finite at } r = 0 \\ \text{ii)} \quad & u = u_s \quad \text{at } r = R(z), \\ \text{iii)} \quad & \frac{\partial u}{\partial r} = 0 \quad \text{at } r = 0, \end{aligned} \quad (2.13)$$

The geometry of the arterial stenosis becomes

$$\begin{aligned} R(z) &= 1 - \delta \left[1 + \cos \left(\frac{\pi z}{2z_0} \right) \right] & \text{if } |z| \leq 2z_0 \\ &= 1 & \text{otherwise} \end{aligned} \quad (2.14)$$

3. Method of Solution:

To solve the non-linear governing equations, the perturbation method is used by considering the pulsatile Reynolds number α , to be small. Since flow is Pulsatile, the non-dimensional governing equations contain the parameter α . Assuming $\alpha^2 \leq 1$, the dependent variables are expanded in a perturbation series in powers of α^2 . Accordingly the axial velocity u , shear stress τ , plug core radius R_p and plug core velocity u_p , wall slip velocity u_s in the form

$$u = u_0 + \alpha^2 u_1 + \dots \quad (3.1)$$

$$\tau = \tau_0 + \alpha^2 \tau_1 + \dots \quad (3.2)$$

$$R_p = R_{0p} + \alpha^2 R_{1p} + \dots \quad (3.3)$$

$$u_p = u_{0p} + \alpha^2 u_{1p} + \dots \quad (3.4)$$

$$u_s = u_{0s} + \alpha^2 u_{1s} + \dots \quad (3.5)$$

Substituting the expansion of u into the equation (2.9) and equating constant term and coefficient of α^2 , we get

$$2(1 + e \cos t) + 2B \cos(\omega t + \varphi) = \frac{1}{2r} \frac{\partial}{\partial r} (r \tau_0) \quad (3.6)$$

$$-\frac{\partial u_0}{\partial t} = -\frac{1}{2r} \frac{\partial}{\partial r} (r \tau_1) \quad (3.7)$$

From (2.11), for $\tau > \tau_H$ we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= -\tau^n \left(1 - \frac{\tau_H}{\tau} \right)^n \\ &= - \left(\tau^n - \tau^{n-1} n \tau_H + \frac{n(n-1)}{2} \tau^{n-2} \tau_H^2 - \dots \right) \end{aligned} \quad (3.8)$$

Substituting the expansion of u and τ from (3.1) in (3.8) and equating constant term and coefficient of α^2 , yield the equation

$$-\frac{\partial u_0}{\partial r} = \tau_0^{n-1}(\tau_0 - n\tau_H) \quad (3.9)$$

$$-\frac{\partial u_1}{\partial r} = n\tau_0^{n-2}\tau_1[\tau_0 - (n-1)\tau_H] \quad (3.10)$$

(3.6) can be written as

$$\frac{\partial}{\partial r}(r\tau_0) = 2rg(t), \quad (3.11)$$

Where $g(t) = 2(1 + e \cos t) + 2B \cos(\omega t + \varphi) \quad (3.12)$

Integrating (3.11) from 0 to R_{0p} gives

$$\tau_{0p} = R_{0p}g(t) \quad (3.13)$$

Integrating (3.11) from R_{0p} to r gives

$$\tau_0 = rg(t) \quad (3.14)$$

Substituting the value of τ_0 from (3.14) into (3.9) and integrating

$$u_0 = (g(t))^n \left[-\frac{r^{n+1}}{n+1} + \frac{r^n \tau_H}{g(t)} \right] + A \quad (3.15)$$

Or alternatively,

$$u_0 = (g(t))^n \left[-\frac{r^{n+1}}{n+1} + r^n q^2 \right] + A, \quad (3.16)$$

Where $q^2 = \frac{\tau_H}{g(t)} \quad (3.17)$

and A is constant of integration.

Substituting the expansions of u and u_s into boundary condition (iii) it is found

$$u_0 = u_{0s} \quad \text{at } r = R \quad (3.18)$$

From (3.15) applying (3.18) we get

$$A = u_{0s} - K(R, t) \quad (3.19)$$

Where $K(r, t) = (g(t))^n r^n \left[q^2 - \frac{r}{(n+1)} \right] \quad (3.20)$

Using Boundary condition (3.18), the velocity u_0 in the outer region is given by

$$u_0 = K(r, t) - K(R, t) + u_{0s} \quad (3.21)$$

The plug core velocity u_{0p} is obtained from eq. (3.21) as

$$u_{0p} = K(R_{0p}, t) - K(R, t) + u_{0s} \quad (3.22)$$

Where $R_{0p} = q^2 = \frac{\tau_H}{g(t)}$

Substituting value of u_0 in (3.7) and integrating we get

$$\tau_1 = -r[K'(r, t) - K'(R, t)], \quad (3.23)$$

where $K'(r, t) = \frac{d}{dt} K(r, t)$

From (3.10) using (3.14), (3.23) we obtained

$$-u_1 = \psi(r, t) + B, \quad (3.24)$$

B constant of integration

$$\begin{aligned} \text{Where } \psi(r, t) = & n(g(t))^{2n-1} \left(\frac{r^{2n+2}}{(n+1)(2n+2)} + \frac{r^{2n+1}\tau_H}{g(t)} \right) - n(g(t))^{2n-2} g'(t) \frac{r^{n+1}}{n+1} \left(\frac{nR^{n+1}}{n+1} + (n-1)R^n\tau_H \right) \\ & - n(n-1)\tau_H(g(t))^{2n-2} \left(\frac{r^{2n+1}}{(n+1)(2n+1)} + \frac{r^{2n}\tau_H}{2ng(t)} \right) + (n-1)\tau_H(g(t))^{3n-3} g'(t) r^n \left(\frac{nR^{n+1}}{n+1} + (n-1)R^n\tau_H \right) \end{aligned} \quad (3.25)$$

Using boundary condition $B = -u_{1s} - \psi(R, t).$

Thus $u_1 = -\psi(r, t) + u_{1s} + \psi(R, t) \quad (3.26)$

And $u_{1p} = -\psi(R_{0p}, t) + u_{1s} + \psi(R, t) \quad (3.27)$

The wall shear stress τ_w is given by

$$\tau_w = [\tau_0 + \alpha^2\tau_1]_{r=R} \quad (3.28)$$

Now $R_{1p} = \frac{(-\tau_1|_{r=R_{0p}})}{g(t)} \quad (3.29)$

and $R_p = R_{0p} + \alpha^2 R_{1p} + ..$

The shear stress $\tau = \tau_0 + \alpha^2\tau_1 \quad \text{at } r = R_p \quad (3.30)$

The non-dimensional volumetric flow rate Q is given by

$$\begin{aligned} Q &= 4 \int_0^{R(z)} ur dr \\ &= 4 \left[\int_0^{R_{0p}} ru_{0p} dr + \int_{R_{0p}}^R ru_0 dr + \alpha^2 \left(\int_0^{R_{0p}} ru_{1p} dr + \int_{R_{0p}}^R ru_1 dr \right) \right] \\ &= 4 \left[\int_0^{R_{0p}} r \left(K(R_p, t) - K(R, t) - \alpha^2\psi(R_p, t) + \alpha^2\psi(R, t) \right) dr + \int_{R_{0p}}^R r \left(K(r, t) - K(R, t) - \alpha^2\psi(r, t) + \alpha^2\psi(R, t) \right) dr + (R^2 - R_{0p}^2)u_s \right] \end{aligned} \quad (3.31)$$

Where we neglect α^4 and higher order and $u_s = u_{0s} + \alpha^2 u_{1s}$. $K(r, t)$ and $\psi(r, t)$ are given by (3.20) and (3.25) respectively.

4. Results and Discussion:

In the present study, a mathematical model has been developed to investigate the effect of wall slip on pulsatile blood flow through a time- dependent stenosed arterial segment, considering blood to be a non-Newtonian fluid and modeled as a Herschel- Bulkley fluid under periodic body acceleration. The governing equations describing the flow dynamics were solved using perturbation analysis, assuming Womersley parameter is small ($\alpha^2 < 1$).

By applying the appropriate boundary conditions, analytical expressions for the velocity profile, wall shear stress and volumetric flow rate were obtained. To evaluate the applicability of the proposed model the variation of velocity (u), flow rate (Q), wall shear stress (τ) were numerically estimated and presented graphically.

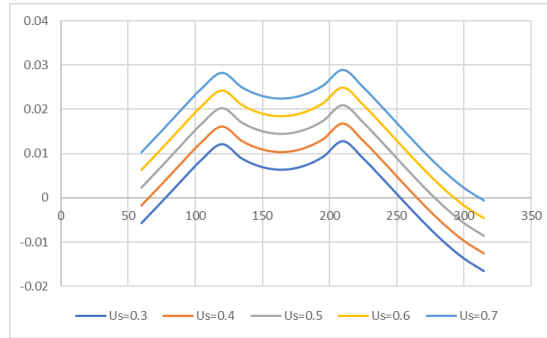


Figure 2 : variation of volumetric velocity flux with time for different value of the wall slip parameter

Figure 2 illustrate the variation of volumetric velocity flux with time for different value of the wall slip parameter u_s . The results indicate that the volumetric flux increases progressively with the increasing wall- slip. For all valued of u_s , the flux exhibits a pulsatile behavior characterized by two distinct peaks during the considered time interval. The peak magnitude increases considerably as the slip parameter increases, demonstrating that wall slip facilitates the transport of blood through the stenosed arterial segment by reducing the effective resistance to flow. Furthermore, an increase in u_s delays the onset of negative volumetric flux, indicating a modification of the flow-reversal characteristics during the pulsatile cycle.

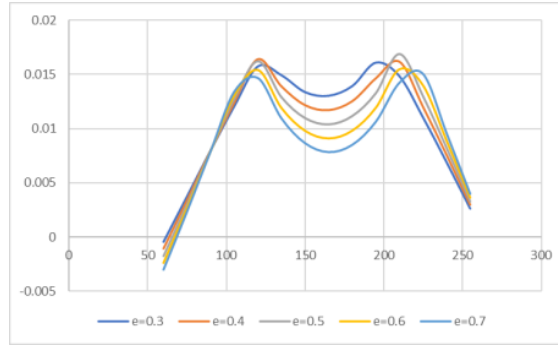


Figure 3 : Variation of volumetric flow rate with time for different values of pressure gradient

The figure 3 illustrate the variation of the volumetric flow rate (Q) with time (t) for different values of the pressure gradient parameter ($e = 0.3 - 0.7$). The volumetric flow rate exhibits a clear pulsatile behaviour with two distinct peak during one cycle, which is characteristic of blood flow under periodic pressure forcing. Around $t \approx 115 - 125$ all curves attain the first maximum value $Q \approx 0.015 - 0.016$. The first peak is only slightly influenced by the pressure gradient parameter, indicating that the initial acceleration phase is relatively insensitive to change in e . Between $t \approx 140$ and $t \approx 180$, the flow rate decreases after the first peak. As e increases, the trough between two peaks becomes deeper, indicating a greater reduction in flow during the deceleration phase. A second peak appear around $t \approx 195 - 210$. Increasing e shift the second peak slightly towards earlier times and slightly towards its magnitude, suggesting that a larger pressure gradient parameter modifies both the timing and amplitude of the pulsatile response. Beyond $t \approx 120$ all curves decreases rapidly and approach nearly the same value by $t \approx 250$, indicating that the influence of e becomes less significant near the end of the cycle. Therefore, influence of pressure gradient is most pronounced during the intermediate phase of the pulsatile flow.

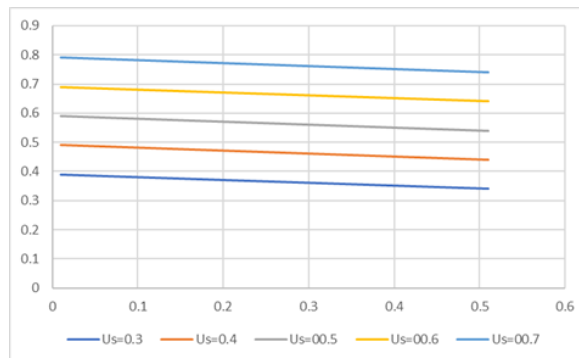


Figure 4 : variations of axial velocity with the plug core radius for different wall slip parameter

Figure 4 illustrates the variations of axial velocity with the radial position in the stenosed arterial segments for different values of the wall slip parameter u_s . It is observed that the axial velocity decreases gradually from the centre line towards the arterial wall for all slip values due to viscous effects. Furthermore, increasing the slip parameter enhances the axial velocity throughout the arterial cross-section. This behaviour occurs because wall slip reduces the share resistance at the vessel wall, thereby facilitating smoother blood motion through the stenosed region. Although velocity

magnitude increases with u_s , the overall shape of the velocity profiles remains nearly unchanged. It is indicating that wall slip primarily influence the flow magnitude rather than the radial distribution.

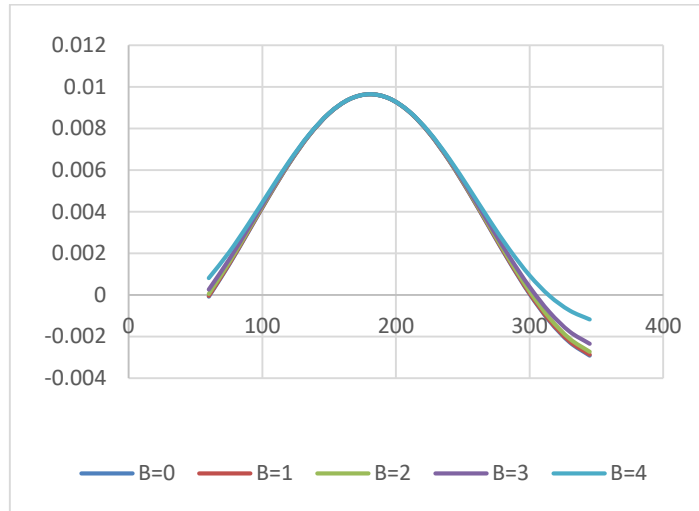


Figure 5 : Variation of volumetric flux rate with time t for different values of body acceleration

Figure 5 illustrates the variation of the volumetric flux Q with time for different values of body acceleration parameter B . The volumetric flux exhibit a pulsatile behaviour, increasing to a maximum around the middle of the cycle before decreasing towards the end. The effect of body acceleration is negligible during the accelerating phase, as the curves nearly coincide and the peak flux remain almost unchanged. However, during the decelerating phase, increasing B slightly reducing the volumetric flux and produces a marginal increase in the reverse flow region. Thus body acceleration primarily influence the late stage of the pulsatile cycle, only a minor effect on the overall flow characteristics.

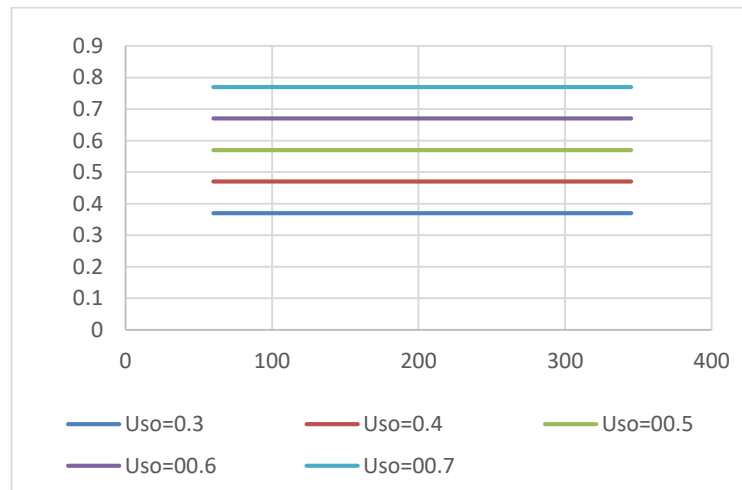


Figure 6 : Variation of axial velocity with time for different values of slip parameter

Figure 6 illustrates the variation of axial velocity u with time t for different values of the wall slip parameter u_s . From the graph it is evident that axial velocity increases with Increasing wall slip parameter. As u_s increases from 0.3 to 0.5 , the velocity profile shift upwards. This indicate that larger wall slip reduces the frictional resistance between the blood and the arterial wall, allowing the fluid move more rapidly. The curves are almost horizontal , which suggests that , under the chosen parameter values , the temporal variation is very small and the flow behaves in a quasi-steady manner. That is, the velocity remain nearly constant with time. The nearly uniform spacing between the successive curves implies that the influence of the wall slip is consistent over the entire time interval. Increasing u_s produces an almost uniform increase in axial velocity without altering the overall temporal trend. The axial velocity remains positive throughout the cycle, indicating that the forward flow is maintained for all considered slip parameter.

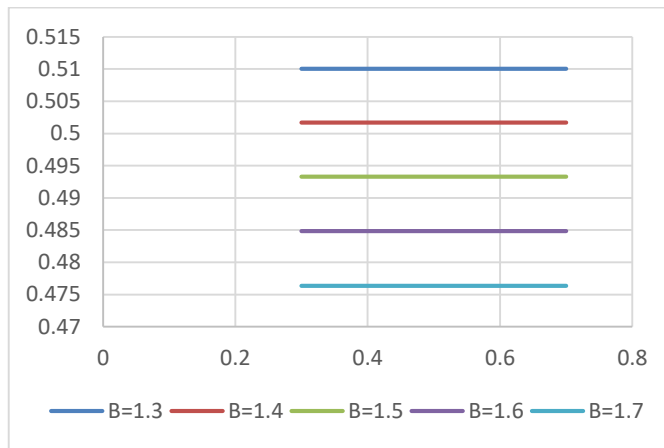


Figure 7 : Variation of volumetric flux with the values of axial length (z) for different value of body acceleration

Figure 7 depicts that the variation of volumetric flux Q along the axial direction (z) for different values of the body acceleration parameter B . It is observed that volumetric flux remains nearly uniform along the axial direction for all values of B . However, the magnitude of Q decreases gradually as the body acceleration parameter increases from 1.3 to 1.7. This indicates that increasing body acceleration slightly suppresses the volumetric flow rate while having a negligible effect on its axial variation. The results suggest that body acceleration primarily influences the overall flow magnitude rather than the special distribution of the flow.

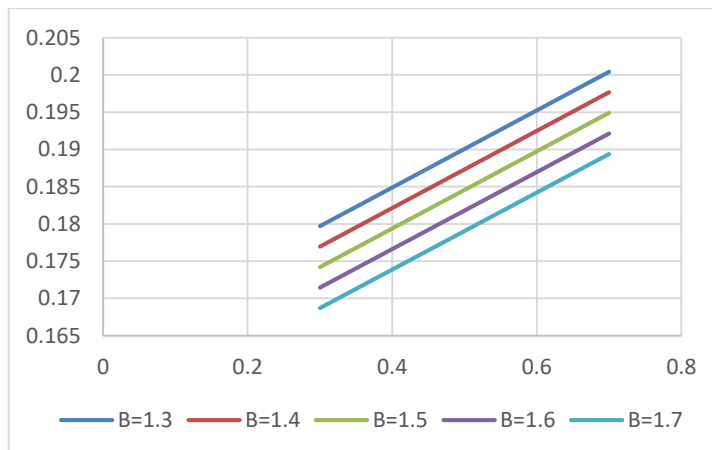


Figure 8 : variation of wall shear stress τ_w with respect to pressure gradient e for different values of body acceleration parameter B

Figure 8 illustrates the variation of wall shear stress τ_w with pressure gradient e for different values of the body acceleration parameter B . It is observed that wall shear stress increases almost linearly with increasing pressure gradient for all values of B .

However, for a fixed pressure gradient, τ_w decreases progressively as the body acceleration parameter increases from $B = 1.3$ to $B = 1.7$.

The nearly parallel nature of curves indicates that the body acceleration primarily reduces the magnitude of the wall shear stress without significantly affecting its sensitivity to the pressure gradient. This result suggested that increasing body acceleration diminishes the viscous shear exerted on the arterial wall while preserving the linear dependence of wall shear stress on the driving pressure gradient.

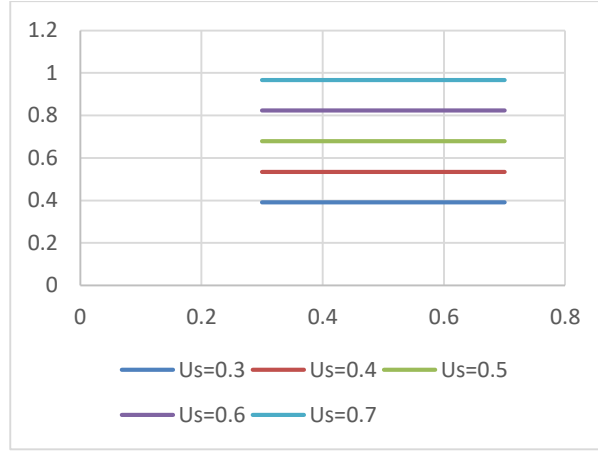


Figure 9 : variation of volumetric flow rate Q with values of z (horizontal axis of the cylinder) for different value of wall slip parameter u_s

Figure 9 illustrates the variation of the volumetric flow rate Q along the axial coordinate z for different value of wall slip parameter u_s . It is observed that the volumetric flow rate increases monotonically with increasing u_s , indicating that wall slip facilitates blood transport by reducing the wall resistance. For each value of u_s , the flow rate exhibits only a slight increase along the axial direction, suggesting the remains nearly uniform throughout the stenosed arterial segment. The nearly parallel nature of the curves further indicates that the influence of wall slip is consistent over the entire computational domain.

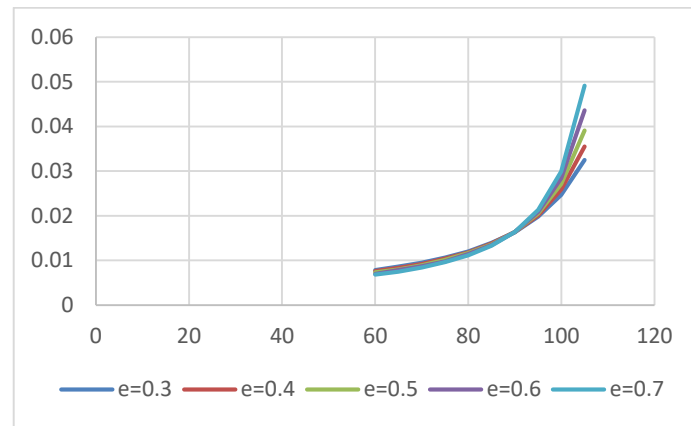


Figure 10. variation of the plug core radius R_p with time for different values of pressure gradient e

Figure 10 illustrates the variation of the plug core radius R_p with time for different values of pressure gradient parameter e . It is observed that the plug core radius increases monotonically throughout the pulsatile cycle, with a relatively slow increase during the early phase followed by a rapid rise near the end of the cycle. The influence of the pressure gradient is insignificant at earlier times, as the curve nearly coincide. However, the differences become more pronounced at latter times where increasing pressure gradient parameter reduces the plug core radius. Thus, a larger pressure gradient suppresses the development of the plug-flow region by increasing the fluid deformation and wall shear effects.

5. Conclusion:

The investigate the influence of wall slip on pulsatile non-Newtonian blood flow through a stenosed artery in presence of the body acceleration. The governing equations were solved to examine the effects of various physiological and flow parameter on the hemodynamic characteristic. The results demonstrate that wall slip significantly modifies the flow behaviour by enhancing the axial velocity and increasing the volumetric flux throughout most of the pulsatile cycle . An increase in the wall slip parameter also delays the onset of flow reversal ,indicating improved transport of blood.

The pressure gradient plays a crucial role in regulating the flow characteristics. An increase in the pressure gradient intensifies the flow ,resulting in higher velocities and altered wall shear stress distributions. The influence of pressure gradient becomes more pronounced during the latter stage of the pulsatile cycles. Furthermore, the plug core radius increases with time but decreases with increasing pressure gradient, suggesting that stronger pressure gradients reduce the extent of the plug -flow region by increasing fluid deformation.

The wall shear stress exhibits marked temporal variations and is strongly influenced by the combined effects of wall slip, body acceleration, and pressure gradient. These findings indicate that wall slip can effectively reduce adverse hemodynamic conditions associated with arterial stenosis by facilitating smoother blood flow while altering the distribution of shear stresses along the arterial wall.

Overall, the study highlights the importance of incorporating wall slip effects in mathematical models of blood flow, particularly in stenosed arteries. The present model provides a better understanding of

the complex interactions among pulsatility, non-Newton behaviour , body acceleration and wall slip, and may contribute to the design of improve diagnostic tools and therapeutic strategies for cardiovascular diseases.

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A Mathematical Study on the Interrelationship between Atmospheric Turbulence and Dispersion of Pollutants

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Abstract:

Atmospheric turbulence is crucial in the spread of pollutants. Essentially, atmospheric turbulence affects the rate of mixing and transport, varying with the intensity and stability of the atmosphere. Greater turbulence intensity leads to reduced pollutant concentrations by enhancing mixing and aiding in the dispersal of pollutants from a location. Conversely, weak turbulence allows pollutants to gather, thereby increasing their concentration close to the surface. In this study, a mathematical model is suggested to grasp the current dynamics to understand the interrelationship between atmospheric turbulence and dispersion of pollutants.

Keynote: Atmospheric turbulence, dispersal of pollutants, transport, intensity.

1. Introduction:

Atmospheric pollution is a significant environmental problem, especially in areas that are growing rapidly in terms of developmental and industrialization activities. Pollution spread depends on the amount of pollution and the complexity of atmospheric turbulence [1, 2, 3, 4], which means the wind is random and unpredictable. If the air is turbulent, the pollution is diluted, meaning the pollution level in that area is lower. The advection-diffusion equations [5, 6] are mathematical tools that help us understand the way in which the pollutant spreads. The equation has two main parts: the way in which the pollutant spreads by means of the average wind speed, and the way in which it spreads by means of the turbulent motion of the air. Calculating how the pollutant spreads is difficult because air turbulence is a constantly changing and random process; it also depends on atmospheric stability, terrain ruggedness, and temperature layering in the air. This study is a way of trying to increase our understanding mathematically, along the way in which these factors affect the spreading of the pollutant.

2. Literature Review:

The way we understand how air turbulence [4] and pollution [3] move through the air has changed significantly in the last century. This evolution is due to advancements in theory, experiments, and computer simulations.

2.1 Theoretical Foundations:

Taylor provided a gateway for in-depth investigation of the phenomenon of spreading things and the concept of turbulent diffusion [6, 7] was thus proposed. This concept is a statistical theory of the motion of fluids. This model is the foundation for understanding the effect of continuous flows on the spreading of pollutants and other substances. In the subsequent years, Richardson entered the scene with observations from the real world and the observation suggested that the degree of dispersion is related to the objects size [8]. This highlighted that turbulence is not a simple phenomenon but a complex one.

2.2 Development of Gaussian Models:

The years spanning the 1930s and 1950s saw the emergence of models known as Gaussian Plume Models [6, 9, 10]. These models operated depending on only one assumption, which is the Gaussian probability function. Despite their simplicity, these have played a crucial role in the field of environmental engineering.

2.3 Advances in Turbulence Theory:

One of the greatest achievements was made by Kolmogorov where he developed a statistical theory of turbulence [11, 12]. The famous $(-\frac{5}{3})$ energy spectrum law [13] helped him to understand the flow of energy in the form of cascade from large scale-dependent [13] turbulence structures to small scale-dependent turbulence structures which helped him to gain deeper insight into the inner structure of turbulent flows. It is still one of the important theories of modern turbulence modelling [4]. Another great development happened in this regard in the form of Monin-Obukhov [14, 15] similarity theory which brought new insights into the phenomenon of turbulence [4].

2.4 Modern Modelling Approaches:

In the recent past, computers have brought about total change in the way we study this field. The use of computer simulation technologies such as Large Eddy Simulations (LES), Reynolds-Averaged Navier-Stokes simulations (RANS) (k - ϵ models and k - ω models), and Stochastic Lagrangian Particle simulations enables us to conduct simulations on how turbulence [4] will influence the dispersion of pollutants. Despite all these advanced computational tools, the simplest analytical solutions of the advection-diffusion equation are

still essential to understand the basic mechanics of this equation. Employing computational methods such as similarity solutions and Green's functions, we have managed to come up with solutions showing basic mechanics such as how the peak pollutant concentration varies inversely.

3. Theory:

Here we consider the advection-diffusion equation [5, 6] which serves as a guide for the dispersion of pollution [3] in environments such as air or water. It looks at a few things: how the fluid itself moves the pollution around, how turbulent mixing randomly spreads it, and if there are any spots where pollution is being added or removed. The transport of pollutant concentration is governed by the following advection-diffusion equation [5, 6]:

$$\partial C / \partial t + \mathbf{U} \cdot \nabla C = \nabla \cdot (\mathbf{K} \nabla C) + S(x, y, z, t) \quad (1)$$

where 'C' is pollutant concentration, 't' is time, and the left side captures temporal change and advection by mean wind velocity 'U'. The first right-hand term models diffusion [6] via the turbulent diffusivity tensor 'K' which depends on atmospheric stability, surface roughness, and turbulence intensity [2, 4], while 'S' represents emissions or removal.

3.1 Advection Mechanism:

The advection mechanism [5] can be thought of as the main current in which the pollution [3] floats. In other words, the fluids average motion (like wind or river flow) moves the pollution. A good analogy for this would be a pile of leaves floating in a river, which would be carried along by the river. In the atmosphere, the main wind currents would be carrying the pollution [16] along, often faster than the diffusion mechanism [6].

3.2 Diffusion Term:

The diffusion [6] term, as described above, concerns how the turbulence [4] in the fluid causes the pollution [3] to spread out. In other words, fluid turbulence [4] stirs the water instead of allowing it to spread out smoothly in all directions (such as the spreading of sugar in water). In the 'K' term, we use a special number that tells us how much and in what directions the fluid turbulence [4] spreads the pollution [3]. Sometimes, turbulence [4] can spread the pollution [3] in one direction rather than another, causing the mixing of the pollution [3] to be much faster than in the diffusion mechanism [6] described in the previous section.

3.3 Source Functionality:

The source or sink functionality of 'S' merely indicates where the pollution [3] is coming from or going to. In other words, if we were modelling a scenario in which there was a chimney from which pollution was emanating, then we would use a positive source. If we were

modelling a scenario in which pollution was settling on the ground (a negative source), then we would use a negative source.

3.4 Turbulence Intensity:

Turbulence intensity [4] is given by

$$I = \frac{u'_{rms}}{U} \quad (2)$$

where u'_{rms} denotes the root-mean-square value of the fluctuating velocity. Higher magnitude of I implies stronger mixing and this further directs to lower concentration. Turbulence intensity [4], as a term, basically tells us how choppy and messy the air or fluid is compared to its normal flow. If turbulence intensity is high, it means there are a lot of intense and swirling eddies. These eddies behave like little blenders, rapidly mixing and spreading out the pollutants, thereby diluting them more quickly, especially near where they were originally emitted.

3.5 Turbulence or Eddy Diffusivity:

Turbulence or Eddy diffusivity term describes how well these eddies transport pollutants vertically, especially under unstable atmospheric conditions [2, 4, 16] (such as when the surface is hotter than the air immediately above it, causing rising air).

$$K = \alpha u_* z \text{ (for unstable boundary layer)} \quad (3)$$

where u_* is friction velocity.

This equation is basically saying that as one goes higher, these eddies mix everything more, largely depending on surface roughness and altitude. This is very important for understanding the rising plumes of pollution [3] from chimneys, for instance.

3.6 Self-Similar Scaling:

Self-similar scaling is basically a pattern that is predictable, where the highest concentration of a pollutant plume decreases as time progresses. This is described by a specific mathematical law, where it decreases very rapidly, specifically as a function of time to the power of negative three-halves, or $t^{-3/2}$. This behaviour is seen when there is consistent turbulence. We have

$$C_{peak}(t) \propto t^{-3/2} \quad (4)$$

which indicates rapid dilution. The equation is basically saying that as time progresses, the plume gets rapidly diluted as it moves further away from where it originally started.

4. Methodology:

4.1 Similarity Transformation:

Turbulent diffusion often exhibits self-similar behaviour, meaning the concentration profiles at different times can collapse onto a single universal curve and the similarity form is [6]

$$C(x, y, z, t) = t^{-3/2} F\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right) \quad (5)$$

This form is physically motivated because in diffusion-dominated systems i) plume width $\sim \sqrt{t}$; ii) peak concentration $\sim t^{-3/2}$ and iii) the variables: $\xi = \frac{x}{\sqrt{t}}, \eta = \frac{y}{\sqrt{t}}, \zeta = \frac{z}{\sqrt{t}}$ collapse the spatial and temporal variation into a single spatial similarity variable. After applying the similarity transformation, (1) takes the following form:

$$\nabla_{\xi, \eta, \zeta}^2 F + \frac{1}{2}(\xi F_{\xi} + \eta F_{\eta} + \zeta F_{\zeta}) + \frac{3}{2}F = 0 \quad (6)$$

To solve (6) by Separation of Variables we consider

$$F(\xi, \eta, \zeta) = X(\xi)Y(\eta)Z(\zeta) \quad (7)$$

Now, applying Green's Function [5, 12] Method for instantaneous point source $S = \delta(x)\delta(y)\delta(z)\delta(t)$

$$\text{we get, } C(x, y, z, t) = \frac{Q}{(4\pi Kt)^{3/2}} \exp\left(-\frac{x^2+y^2+z^2}{4Kt}\right) \quad (8)$$

From (8) we observe that the diffusion behaviour exhibits $t^{-3/2}$ scaling.

4.2 Advection-diffusion with constant eddy diffusivity:

We consider an instantaneous point release mass 'M' using a one-dimensional advection - diffusion, and also, we consider a constant mean wind U and effective diffusivity K. Then from (1) we have

$$\frac{\partial c}{\partial t} + U \cdot \frac{\partial c}{\partial x} = K \frac{\partial^2 c}{\partial x^2}, \quad C(x, 0) = M\delta(x) \quad (9)$$

Hence the solution (8) can be written in the form of Gaussian Function or Converted Gaussian [6, 9, 10] solution as,

$$C(x, t) = \frac{M}{\sqrt{4\pi Kt}} \exp(-(x-Ut)^2) \quad (10)$$

(10) expresses how the puff spreads proportionally with $\sqrt{Kt}$ while its centre advects at speed U. This classic result follows directly from methods of Fourier transforms or similarity transforms and is based on Taylor's diffusion [6] concept.

4.3 Instantaneous Point, Anisotropic Diffusivities, and Steady Wind function:

After considering instantaneous release of mass M at $x=0$ and $t=0$, when the puff centre moves at $x=Ut$, then the spread is $\delta_x = \sqrt{2Kt}$, and peak concentration decreases as $\frac{1}{\sqrt{t}}$. And also, if we consider a point release of total mass M at the source location (0, 0, K) at time $t=0$, the mean wind takes the form $U=(U,0,0)$ and constant turbulent diffusivities K_x, K_y , and K_z possibly

differ in three directions. Taking molecular diffusion negligible compared to the turbulent diffusion [6] the concentration $C(x, y, z, t)$ at (x, y, z) and time $t > 0$ can be expressed as

$$C(x, y, z, t) = \frac{M}{(4\pi t)^{\frac{3}{2}} \sqrt{K_x K_y K_z}} \exp\left(-\frac{(x-Ut)^2}{4K_x t} - \frac{y^2}{4K_y t} - \frac{(z-H)^2}{4K_z t}\right) \quad (11)$$

which is known as the fundamental (Green's function) solution of the advection-diffusion [4, 7] equation with constant coefficients. If we need the ground reflection (i.e., no flux through $z=0$ or an impermeable ground), we add the image term with $z \rightarrow (-H)$ and then

$$C_{\text{ground}}(x, y, z, t) = C(x, y, z, t) + C(x, y, -z, t) \text{ (i.e. image source)} \quad (12)$$

4.4 Instantaneous advection–diffusion Partial Differential Equation (for incompressible):

We express instantaneous advection-diffusion [5, 6] equation for incompressible fluid as

$$\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} = K_x \frac{\partial^2 C}{\partial x^2} + K_y \frac{\partial^2 C}{\partial y^2} + K_z \frac{\partial^2 C}{\partial z^2} \quad (13)$$

with initial condition at $C(x, y, z, 0) = M \delta(x) \delta(y) \delta(z - H)$, and we also define $\xi = x - Ut$.

Then $\partial/\partial t|_x = \partial/\partial t|_{\xi} - U \partial/\partial \xi$, and $\partial/\partial x = \partial/\partial \xi$ and from (13) we have

$$\left(\frac{\partial C}{\partial t}\right)_{\xi} - U \frac{\partial C}{\partial \xi} + U \frac{\partial C}{\partial \xi} = K_x \frac{\partial^2 C}{\partial \xi^2} + K_y \frac{\partial^2 C}{\partial y^2} + K_z \frac{\partial^2 C}{\partial z^2} \quad (14)$$

Now cancelling the advection term, we get a pure diffusion Partial Differential Equation in the convected frame:

$$\left(\frac{\partial C}{\partial t}\right)_{\xi} = K_x \frac{\partial^2 C}{\partial \xi^2} + K_y \frac{\partial^2 C}{\partial y^2} + K_z \frac{\partial^2 C}{\partial z^2} \quad (15)$$

4.5 The fundamental solution of the anisotropic heat equation:

The product of three independent Gaussian kernels [6, 9, 10] gives the fundamental solution for anisotropic constant-coefficient diffusion [6] with initial stage at $(\xi, y, z) = (0, 0, H)$;

Now substituting $x = \xi + Ut$ in (10), we obtain the following:

$$C(\xi, y, z, t) = \frac{M}{(4\pi t)^{\frac{3}{2}} \sqrt{K_x K_y K_z}} \exp\left(-\frac{\xi^2}{4K_x t} - \frac{y^2}{4K_y t} - \frac{(z-H)^2}{4K_z t}\right) \quad (16)$$

Now, from (15) we get,

Centre of mass (mean position): $x_c(t) = Ut$, $y_c = 0$, $z_c = H$ (puff centre advects with mean wind).

Directional spreads (standard deviations):

$$\sigma_x(t) = \sqrt{2K_x t}, \quad \sigma_y(t) = \sqrt{2K_y t}, \quad \sigma_z(t) = \sqrt{2K_z t} \quad (17)$$

These quantify the typical width of the puff in each direction.

Peak (centreline) concentration at the puff centre ($x = Ut, y = 0, z = H$):

$$C_{\text{peak}}(t) = \frac{M}{(4\pi t)^{\frac{3}{2}} \sqrt{K_x K_y K_z}} \quad (18)$$

As time increases, $C_{\text{peak}} \propto t^{-3/2}$ (i.e. rapid dilution). With that we have the following:

i. Initial Condition: $C(x, y, z, 0) = M\delta(x)\delta(y)\delta(z - H)$

ii. Ground Reflection: $\frac{\partial C}{\partial z} = 0 \quad (z = 0)$

iii. Far-Field Conditions: $C \rightarrow 0 \text{ as } x, y, z \rightarrow \infty$

Here these three conditions are sufficient to derive the classical Gaussian and Green's function solutions which are discussed in this section.

5. Conclusion:

This advection-diffusion [5, 6] approach provides a realistic description of the transport of pollutants in the atmosphere. The advection [5] term controls the downstream transport of the plume with the mean wind velocity. On the other hand, the effect of turbulence [4] is included in the diffusion [6] term. An increase in the intensity of turbulence causes the diffusivity of the turbulent eddies to rise. This causes the plume width to increase and the maximum concentration of the plume to decrease. This demonstrates the effect of turbulence in the dispersion [3] of pollutants. On the other hand, a weak turbulence [4] intensity causes the vertical motion of the plume to decrease, resulting in a confined plume with a high concentration of pollutants in the near-surface region. The self-similar properties of the solutions show that the width of the plume increases with time while the maximum concentration of the plume decreases with time in a power-law manner.

The analytical solutions [5, 12], especially those based on the Gaussian [6, 9, 10] and Green's functions, effectively illustrate the change in the concentrations of pollutants depending on the turbulence levels. The research confirms that increased turbulence accelerates the spread of pollutants, while stable conditions impede their dispersion. This framework not only provides a deeper understanding of the effects of turbulence on the spread of matter but also provides a solid foundation for the modelling of better air quality and environmental forecasting. These mathematical descriptions are fundamental in linking the existing turbulence theories with the current computational approaches.

6. Future Scopes:

Future study may incorporate the following fields: i) Integration with stochastic Lagrangian particle models; ii) Extension to urban canopies and complex terrain; iii) Coupling pollutant dispersion with chemistry and aerosol microphysics; iv) Application of Large Eddy Simulation (LES) and Direct Numerical Simulation (DNS) data to refine parameterizations.

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