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Inflation-a Comparative Study Amongst Different Modified Gravity Theories.

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[**Abstract:** In the recent years, a host of modified gravity models have been proposed as alternatives to the dark energy. A quantum theory of gravity also requires to modify `General Theory of Relativity'. In the present article, we consider five different modified theories of gravity, and compare inflationary parameters with recent data sets released by two Planck collaboration teams. Our analysis reveals that the scalar-tensor theory of gravity is the best alternative].

Key wards: *Inflation, Graceful exit, Modified theory of gravity, Matter dominated era.*

1. Introduction

After some initial debate, cosmologists have unanimously and unambiguously come to a very weird conclusion that the universe is currently accelerating. Weird, since gravity is attractive, a fifth force (quintessence) must be responsible for such a phenomenon. General Theory of Relativity (GTR) described by the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu} \qquad \ldots (1)$$

where left hand side is the Einstein tensor which describes the curvature

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of space time, and the right hand side is the energy-momentum tensor of baryonic matter and non-baryonic dark matter, $\kappa = 8\pi G, G$ being the Newton's gravitational constant; cannot address such phenomena. The reason being: the equation of state parameter is $\omega = \frac{p}{\rho} \geq 0$ (where p and ρ are the thermodynamic pressure and matter density respectively), while accelerated expansion of the universe requires a negative pressure, so that the equation of state parameter is $\omega_e < -\frac{1}{3}$, where, the subscript `$e'$ stands for `effective'. To be precise, current data suggests $\omega_e < -\frac{2}{3}$. Therefore, GTR somehow, has to be modified. Cosmological constant (Λ), for which $\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1,$ can resolve the issue single handedly, but then, what is a cosmological constant? A physical interpretation of it comes from high energy physics, in which one can compute a `constant' available in the nature, as the sum of vacuum energy densities of all types of matter existing in the universe. Unfortunately, the constant required for current acceleration of the universe is 120 order of magnitude smaller than the sum of vacuum energy densities. Thus, ΛCDM (cold dark matter) model was replaced initially by a quintessence field, which is essentially a scalar field, for which

$$\omega_e = \frac{p_e}{\rho_e} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}.$$

Clearly, quintessence model does not admit the value of the equation of state parameter to go beyond the phantom divide line, $\omega = -1,$ since if, $\dot{\phi}^2 \ll V(\phi),$ then $\omega_e = -1.$ However, crossing of the phantom divide line is not excluded by observations. Therefore, different exotic models (K-essence, Tachyon, holographic model etc.) were proposed. These are all dark energy models, since such fields interact none other than gravity itself. These fields essentially modify the energy-momentum tensor ($T_{\mu\nu}$), that is the right hand side of Einstein's equation of GTR. However, since all attempts to detect dark energy has failed(There is a very recent indication of direct detection of dark energy in XENON1T, that we shall discuss in brief in the conclusion), so cosmologists started modifying the left hand side of Einstein's equation, namely the curvature part. Einstein's equation of GTR(1) may be found under the variation of the so-called Einstein-Hilbert action,

$$A = \int \left[\frac{R}{16\pi G}\right]\sqrt{-g}\, d^4x + S_m, \qquad \ldots (2)$$

where, R is the Ricci scalar, $-g$ is the determinant of the metric, and S_m is the matter action. In order to modify the left hand side of Einstein's equation (1), it is required to replace Ricci scalar (R) by a generalized curvature scalar. A host of such models $F(R), F(\mathcal{G}), F(T), F(Q)$ etc., where, $\mathcal{G} = R^2 + 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\delta\gamma}R^{\mu\nu\delta\gamma}$ is the Gauss-Bonnet term, T is the torsion term, Q is the non-metricity scalar, have so far been proposed. All these models can address late-time cosmic acceleration followed by an early decelerating phase. On the other hand, construction of a quantum theory of gravity also requires to modify GTR, by incorporating higher order curvature invariant terms in the gravitational action[1]. It is therefore suggestive to check if these modified theories can explain early stage of cosmic evolution, an inflationary phase, in particular.

Standard model of cosmology, the so-called `Friedmann-Lemaitre-Robertson-Walker (FLRW) model' predicts that the universe initiated from a big-bang, represented by a hot thick soup of plasma. The evidence of extremely hot big-bang has been experimentally verified through the detection of CMBR (Cosmic Microwave Background Radiation). However, causally disconnected regions appear to be isotropic up to 10^{-5} order of magnitude, called the horizon problem, which is not explained by FLRW model. Further, FLRW model does not explain flatness problem (the fact that the universe is almost flat at present, and a slightest deviation, would have collapsed it very early, or would have enormously expanded it, desisting to form structures). Finally, FLRW model does not also account for the structure (stars, galaxies, cluster of galaxies etc.) formation. All these issues may be addressed if there had been a stage of inflation (exponential or power law expansion of the scale factor $a(t)$) in the very earlystage of cosmological evolution[2 – 6]. Although, there exists some models which appear to explain these issues[7–9,] inflation is prevalent, mainstream choice, and is considered to be a scenario, rather than a model. In this connection, it is suggestive to check, if proposed modified theories of gravity can accommodate inflation as well. Inflation is essentially a quantum phenomenon, which was initiated sometime between (10^{-42} to $10^{-26}\ s$), after gravitational sector transits to the classical domain. To be more specific, it is a quantum theory of perturbations on top of a classical background, which means the energy scale of the background must be much below Planck scales. There is also recent evidence from the string theory swampland that the energy scale must be rather low for inflation. Despite the fact that inflation is a quantum phenomenon, most of the important physics may be extracted from the classical action itself, provided the quantum theory admits a viable semi-classical approximation. We have shown earlier that the models under consideration, admit viable quantum dynamics and are classically allowed, since the semi-classical wave-functions oscillate about classical inflationary solutions.

In view of the above discussions, in the following section of the present article, we consider five different well versed modified theories of gravity to study inflation. In particular, we inspect how far these models fit with the currently released inflationary parameters[10, 11], namely the tensor to scalar ratio $r = 16\epsilon < 0.06$, where ϵ isthe first slow roll parameter, and the scalar tilt, or more conventionally the spectral index of scalar perturbation $0.096 < n_s < 0.097$. Further, the number of e-folding should remain preferably within the range $40 < N < 70$, to solve the horizon and the flatness problems. First four of these models are higher order theories, while one appearing at the end, is a non-minimally coupled scalar-tensor theory of gravity. The comparative study that we are going to perform, will render a selection rule to consider a particular modified gravitational action. In section3 we conclude.

2. Inflation in different modified theories of gravity

So far, all attempts to cast a viable (although unsuccessful) quantum theory of gravity addressed higher order scalar curvature invariant terms ($R^2,\ R_{\mu\nu}R^{\mu\nu}$)) in the action. Further, Gauss-Bonnet term $\mathcal{G} = \left(R^2 + R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\delta\gamma}R^{\mu\nu\delta\gamma}\right)$, appears quite naturally as the leading order of the inverse string tension α' expansion of heterotic superstring theory[12–15]. But, Gauss-Bonnet term is topologically invariant in 4-dimension, which means, it is a total derivative term, and therefore does not contribute to the field equations. However, when coupled to a scalar field (dilation), it contributes. In this context, it is noteworthy that the low energy limit of the string theory gives rise to the dilatonic scalar field, which is also found to be coupled with various curvature invariant terms[16,17]. Therefore, the leading quadratic correction gives rise to Gauss-Bonnet term with a dilatonic coupling[18]. It is important to mention that the dilatonic coupled Gauss-Bonnet term plays a vital role at the late-stage of cosmic evolution (pressure-less dust era), exhibiting accelerated expansion after a long Friedmann-like decelerating phase[19, 20]. The higher order theories under consideration therefore should contain these terms in different combinations.

We shall work in the following homogeneous and isotropic Robertson-Walker metric, viz.,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \qquad \dots(3)$$

where, $a(t)$ is the scale factor. The Ricci scalar and the Gauss-Bonnet term for the above space-time (3) are givenby,

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \qquad \mathcal{G} = 24\frac{\ddot{a}}{a^3}(\dot{a}^2 + k), \qquad \dots(4)$$

which we shall require to cast the field equations.

2.1 Case-1

First, we start with the following generalized action considered earlier[21].

$$A_1 = \int d^4x\sqrt{-g}\left[\alpha(\phi)R - \Lambda M_P^2 + \beta(\phi)R^2 + \gamma(\phi)\mathcal{G} - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right] \qquad \dots(5)$$

The action contains undetermined coupling parameters $\alpha(\phi)$, $\beta(\phi)$, and $\gamma(\phi)$ and a cosmological constant term(Λ) being coupled to the reduced Planck′s mass $M_P^2 = \frac{1}{8\pi G}$.

2.1.1 Field equations and classical solutions

The field equations, namely the `a' variation i.e. $\binom{i}{i}$equation, the $\binom{0}{0}$ equation and the ϕ variation equation for the metric (3) are the following,

$$2\alpha\left[\frac{2\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right]+2\alpha'\left[\ddot{\phi}+\frac{2\dot{a}\dot{\phi}}{a}\right]+2\alpha''\dot{\phi}^2$$
$$+12\beta\left[\frac{2\dddot{a}}{a}+\frac{4\dot{a}\dddot{a}}{a^2}+\frac{3\ddot{a}^2}{a^2}-\frac{12\dot{a}^2\ddot{a}}{a^3}+\frac{3\dot{a}^4}{a^4}-\frac{4k\ddot{a}}{a^3}\right.$$
$$\left.+\frac{2k\dot{a}^2}{a^4}-\frac{k^2}{a^4}\right]+48\beta'\dot{\phi}\left[\frac{\dddot{a}}{a}+\frac{2\dot{a}\ddot{a}}{a^2}-\frac{\dot{a}^3}{a^3}-\frac{k\dot{a}}{a^3}\right]$$
$$+\left(24\beta''\dot{\phi}^2+24\beta'\ddot{\phi}\right)\left[\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right]+\frac{16\gamma'\ddot{a}\dot{a}\dot{\phi}}{a^2}$$
$$+8\gamma'\ddot{\phi}\left[\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right]+8\gamma''\dot{\phi}^2\left[\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right]+\frac{\dot{\phi}^2}{2}-V-\Lambda M_P^2$$
$$=0 \qquad \ldots(6)$$

$$6\alpha\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)+6\alpha'\dot{\phi}\left(\frac{\dot{a}}{a}\right)$$
$$+36\beta\left(\frac{2\dot{a}\dddot{a}}{a^2}-\frac{\ddot{a}^2}{a^2}+\frac{2\dot{a}^2\ddot{a}}{a^3}-\frac{3\dot{a}^4}{a^4}-\frac{2k\dot{a}^2}{a^4}+\frac{k^2}{a^4}\right)$$
$$+72\beta'\dot{\phi}\left(\frac{\dot{a}\ddot{a}}{a^2}+\frac{2\dot{a}^3}{a^3}+\frac{k\dot{a}}{a^3}\right)+24\gamma'\dot{\phi}\left(\frac{\dot{a}^3}{a^3}+\frac{k\dot{a}}{a^3}\right)-\Lambda M_P^2$$
$$=\left(\frac{\dot{\phi}^2}{2}+V\right) \qquad \ldots(7)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V' - 6\alpha'\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) - 36\beta'\left(\frac{\ddot{a}^2}{a^2} + 2\frac{\dot{a}^2\ddot{a}}{a^3} + \frac{\dot{a}^4}{a^4} + 2\frac{k\ddot{a}}{a^3} + \frac{2k\dot{a}^2}{a^4} + \frac{k^2}{a^4}\right) - 24\gamma'\left(\frac{\dot{a}^2\ddot{a}}{a^3} + \frac{k\ddot{a}}{a^3}\right) = 0 \quad \ldots (8)$$

In the above, and throughout, an over-dot denotes time derivative, while prime denotes derivative with respect to the scalar field. Not all the above components of Einstein's equations are independent, since the $\binom{0}{0}$ equation is the energy constraint equation. Thus it suffices to consider only the two independent components of Einstein's equations, viz. (7) and (8), for all practical purposes. A viable gravity theory must admit de-Sitter solution($\alpha \propto e^{\lambda t}$) in vacuum. As already explored earlier[21], the above field equations admit the following de-Sitter solution in the spatially flat space ($k = 0$) space-time,

$a = a_0 e^{\lambda t};\ \phi = \phi_0 e^{-\lambda t},$ under the condition;

$$\alpha(\phi) = \frac{\alpha_0}{\phi};\ V(\phi) = \frac{1}{2}\lambda^2\phi^2 - \Lambda M_P^2; \text{and } 6\beta(\phi) + \gamma(\phi) = -\frac{1}{2\lambda^2}\left(\frac{\alpha_0}{\phi} + \frac{\phi^2}{24}\right) \quad \ldots (9)$$

where, a_0, ϕ_0, α_0 and λ are arbitrary constants, while $\beta(\phi)$ and $\gamma(\phi)$ remain arbitrary functions of ϕ, being related as above, after setting the constant of integration to zero without any loss of generality.

2.1.2 Inflation under Slow Roll Approximation

As mentioned in the introduction, the model under consideration admits a viable (Hermitian) quantum dynamics, while the semi-classical wave-function oscillates about the above classical inflationary solution

(9), and thus it is classically allowed. Hence although Inflation is a quantum mechanical phenomena, most of the important physics are inherent in the classical action. We therefore proceed to study inflation and see how far the inflationary parameters viz. the tensor to scalar ratio r and the spectral index of scalar perturbation n_s fit with currently released data sets $r < 0.06$ and $0.096 < n_s < 0.097$, keeping the number of e-folding within the range $45 < N < 70,$ required to solve the horizon and the flatness problems[10, 11]. For a complicated theory such as the present one, it is of course a very difficult task. However, we followed a unique technique to make things look rather simple, which is described underneath. We express equations (7) and (8) in terms of the Hubble parameter H in the spatially at space-time ($k = 0$) as,

$$6\alpha H^2 = \frac{\dot{\phi}^2}{2} + [V + \Lambda M_P^2 - \left(6\alpha'\dot{\phi}H + 144\beta\dot{\phi}H^3 + 24\gamma'\dot{\phi}H^3\right] \quad \ldots (10)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + [V' - (12\alpha' H^2 + 144\beta' H^4 + 24\gamma' H^4] = 0 \quad \ldots (11)$$

Above equations (10) and (11) are still formidably complicated to handle, and so before imposing the standard slow roll conditions, viz. $|\ddot{\phi}| \ll 3H|\dot{\phi}|$ and $\dot{\phi}^2 \ll V(\phi)$, further simplification is required. One way is to use additional hierarchy of flow parameters[22-25] in connection with additional degrees of freedom associated with the present model. Instead, we shall follow a completely different and unique technique. For example, redefining the potential as,

$$U = V - 12H^2(\alpha + 12H^2\beta + 2H^2\gamma), \quad \ldots (12)$$

equation (11) takes the following form of the standard Klein-Gordon equation,

$$\ddot{\phi} + 3H\dot{\phi} + U' = 0 \quad \ldots (13)$$

Clearly the evolution of the scalar field is driven by the re-defined potential gradient $U' = \frac{dU}{d\phi}$, subject to the damping by the Hubble expansion $3H\dot{\phi}$, as in the case of single field equation coupled to Einstein-Hilbert term. Note that the potential $U(\phi)$ carries all the information in connection with the coupling parameters of generalised higher order action under consideration. Further, assuming

$$U = V + \Lambda M_P^2 - 6H\dot{\phi}(\alpha' + 24H^2\beta' + 4H^2\gamma') \quad \dots (14)$$

equation (10) may be reduced to the following simplified form, viz.

$$6\alpha H^2 = \frac{\dot{\phi}^2}{2} + U(\phi) \quad \dots (15)$$

which is essentially the non-minimally coupled Einstein's $\binom{0}{0}$ equation. It is noteworthy that, the above two choices (12) and (14) of $U(\phi)$, do not contradict, since the two simply results in an evolution equation of the scalar field

Table–1

α_0 in M_P^3	ϕ_f in M_P	n_s	r	N
0.00036	4.88810	0.9693	0.08322	49
0.00037	4.88822	0.9684	0.08553	48
0.00038	4.88833	0.9676	0.08784	47
0.00039	4.88845	0.9667	0.09016	45
0.00040	4.88856	0.9659	0.09247	44
0.00041	4.88868	0.9650	0.09478	43
0.00042	4.88879	0.9642	0.09709	42
0.00043	4.88890	0.9633	0.09940	41

Data set for the inflationary parameters taking $\phi_i = 5.0M_P; m^2 = 0.084M_P^2$; $\Lambda = 1M_P^2$ and varying α_0, keeping n_s within Planck's constraint limit.

Table–2

$\alpha_0$in M_P^3	ϕ_f in M_P	n_s	r	N
0.000240	4.88652	0.9795	0.05548	74
0.000242	4.88655	0.9793	0.05594	74
0.000244	4.88658	0.9792	0.05640	73
0.000248	4.88663	0.9788	0.05733	72
0.000252	4.88669	0.9785	0.05825	71
0.000256	4.88675	0.9782	0.05918	70
0.000258	4.88678	0.9780	0.05964	69

Data set for the inflationary parameters taking $\phi_i = 5.0M_P; m^2 = 0.84M_P^2$; $\Lambda = 1M_P^2$ and varying α_0, keeping n_s within Planck's constraint limit.

ϕ, which may be different from (9), since during inflation the Hubble parameter is slowly varying. Shortly, we shall show that ϕ indeed falls with time, which has already been demonstrated in[21]. Now, let us enforce the standard slow-roll conditions $\dot{\phi}^2 \ll U$ and $|\ddot{\phi}| \ll 3H|\dot{\phi}|$, so that equations (15) and (13) finally reduce to,

$$6\alpha H^2 \simeq U, \qquad \text{... (16)}$$

and

$$3H\dot{\phi} \simeq U', \qquad \text{... (17)}$$

respectively. Combining equations (16) and (17), it is possible to show that the potential slow roll parameter ϵ equals the Hubble slow roll parameter (ϵ_1) under the condition,

$$\epsilon = -\frac{\dot{H}}{H^2} = \alpha\left(\frac{U'}{U}\right)^2 - \alpha'\left(\frac{U'}{U}\right);\ \eta = 2\alpha\left(\frac{U''}{U}\right) \qquad \text{... (18)}$$

Clearly, for α = constant, the second term vanishes and the standard relation is restored, while the other slow-roll parameter η remains unaltered. Further since, $\frac{H}{\dot{\phi}} = -\frac{U}{2\alpha U'}$ in view of equations (16) and (17), therefore,the number of e-folds (N) at which the present Hubble scale equals the Hubble scale during inflation, may be computed as usual in view of the following relation:

$$N(\phi) \simeq \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi = \int_{\phi_f}^{\phi_i} \left(\frac{U}{2\alpha U'}\right) d\phi, \qquad \text{... (19)}$$

where, ϕ_i and ϕ_f denote the values of the scalar field at the beginning (t_i) and the end (t_f) of inflation. Thus, slow roll parameters reflect all the interactions, as exhibited earlier[23, 26], but here only via the redefined potential $U(\phi)$. Now, during inflation the Hubble parameter remains almost constant, and therefore while computing $U(\phi)$, one can replace it by the constant λ, without any loss of generality. Thus, using classical solutions (9), we can express (12) as,

$12H^2(\alpha + 12H^2\beta + 2H^2\gamma) \approx \frac{H^2\phi^2}{2}$, such that, $U = \frac{1}{2}m^2\phi^2 - \Lambda M_P^2$, where, $m^2 = \lambda^2 + H^2 \approx 2\lambda^2$. ... (20)

Hence, the slow roll parameters along with the number of e-folds, read as,

$$\epsilon = \frac{4m^4\alpha_0\phi}{(m^2\phi^2 - 2\Lambda M_P^2)^2} + \frac{2m^2\alpha_0}{(m^2\phi^3 - 2\phi\Lambda M_P^2)},$$

$$\eta = \frac{4m^2\alpha_0}{m^2\phi^3 - 2\phi\Lambda M_P^2} \qquad \text{... (21)}$$

$$N = \frac{1}{4\alpha_0}\int_{\phi_f}^{\phi_i} \left(\frac{m^2\phi^2 - 2\Lambda M_P^2}{m^2}\right) d\phi$$

$$= \frac{1}{12\alpha_0}\left(\phi_i^3 - \phi_f^3\right) - \frac{\Lambda M_P^2}{2m^2\alpha_0}\left(\phi_i - \phi_f\right) \qquad \text{... (22)}$$

Taking $\phi_i = 5M_P,\ m^2 = 0.084M_P^2$ and $\Lambda = 1M_P^2$, we exhibit our data sets in a pair of tables 1 and 2, and find that inflation ends ($\epsilon = 1$) at around $\phi_f \approx 4.49M_P$.$\alpha_0$is varied differently in the two tables to keep n_s

within the experimental limit in the first, and r within the experimental limit in the second. In the first case, we see that it is not possible to reduce r below 0.08, while the second table depicts that n_s exceeds the experimental limit. Of course, the Planck's data might vary a little for different models. In this respect, the fit is fair.

To show the consistency of our choice of redefined potential presented in equations (12) and (14), we combine the two, to obtain the following first order differential equation on ϕ,

$$\left(\frac{\phi^3 - 6\alpha_0}{\lambda^2\phi^4 - 2\Lambda M_P^2\phi^2}\right)d\phi = \frac{1}{2\lambda}dt. \qquad \ldots (23)$$

Although the above differential equation may be solved exactly, it is extremely difficult to study its nature. We therefore neglect the second term in the numerator with respect to the first and the first term in the denominator with respect to the second, in view of our data (table-1 and table-2). Under such approximation equation (23)may be expressed as:

$$\dot{\phi} \approx -\left[\frac{M_P^2\Lambda}{\lambda\phi}\right] \qquad \ldots (24)$$

Clearly, ϕ falls with time. Additionally, it may be mentioned that the energy scale of inflation has been found to be sub-Planckian[21] $H_* \approx 10^{-5}M_P$. Further, the model admits graceful exit from inflation, since the scalar field starts oscillating, $\phi \sim \pm\frac{\sqrt{\Lambda}M_P}{\sqrt{2\Lambda}}\sin\left(\sqrt{2}\lambda t - \sqrt{2}c_1\lambda\right)$, many times over a Hubble time, driving a matter-dominated era at the end ofinflation[21].

2.2 Case-2

Next, we consider an even more general action explored in[26], which is the following,

$$A = \int \left[\alpha(\phi)R + \beta_1(\phi)R^2 + \beta_2(\phi)\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) + \gamma(\phi)\mathcal{G} - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right] d^4x\sqrt{-g}. \quad \ldots(25)$$

Note that here we consider the additional curvature squared term, viz. $R_{\mu\nu}^2$, with an additional ϕ dependentcoupling parameter $\beta_2(\phi)$.

2.2.1 Field equations and classical solutions

Due to diffeomorphic invariance (energy constraint), only two components of Einstein's equations are independent, as mentioned previously. We therefore consider the $\binom{0}{0}$ and the ϕ variation equations in the background of Robertson-Walker metric (3), which are,

$$-\frac{6\alpha}{a^2}(\dot{a}^2 + k) - \frac{6\alpha'\dot{a}\dot{\phi}}{a} - 36\beta_1\left(\frac{2\dot{a}\dddot{a}}{a^2} - \frac{\ddot{a}^2}{a^2} + \frac{2\dot{a}^2\ddot{a}}{a^3} - \frac{3\dot{a}^4}{a^4} - \frac{2k\dot{a}^2}{a^4} + \frac{k^2}{a^4}\right) - 72\beta_1'\dot{\phi}\left(\frac{\dot{a}\ddot{a}}{a^2} + \frac{\dot{a}^3}{a^3} + \frac{k\dot{a}}{a^3}\right) + 6\beta_2'\dot{\phi}\left(\frac{2\dot{a}^3}{a^3} + \frac{3k\dot{a}}{a^3}\right) -24\gamma'\dot{\phi}\left(\frac{\dot{a}^3}{a^3} + \frac{k\dot{a}}{a^3}\right) + \left(\frac{\dot{\phi}^2}{2} + V\right) = 0 \quad \ldots(26)$$

and

$$-6\alpha'(a^2\ddot{a} + a\dot{a}^2 + ka) - 36\beta_1'\left(a\ddot{a}^2 + 2\dot{a}^2\ddot{a} + \frac{\dot{a}^4}{a} + \frac{k^2}{a} + \frac{2k\dot{a}^2}{a} + 2k\ddot{a}\right) + 12\beta_2'(\dot{a}^2\ddot{a} + k\ddot{a}) + 3a^2\dot{a}\dot{\phi} - 24\gamma'(\dot{a}^2\ddot{a} + k\ddot{a}) + a^3(\ddot{\phi} + V') = 0 \quad \ldots(27)$$

The above field equations also admit the following de-Sitter solution in the spatially at space ($k = 0$),

$a = a_0 e^{\lambda t}; \phi = \phi_0 e^{-\lambda t}$, under the condition,

$$\alpha(\phi) = \frac{\alpha_0}{\phi}; \quad V(\phi) = \frac{1}{2}\lambda^2\phi^2; \text{ and } \beta_2 - 2(6\beta_1 + \gamma) = \frac{1}{\lambda^2}\left(\frac{\alpha_0}{\phi} + \frac{\phi^2}{24}\right) \quad \ldots(28)$$

where, a_0, ϕ_0, α_0 and λ are arbitrary constants while $\beta_1(\phi)$, $\beta_2(\phi)$ and $\gamma(\phi)$ remain arbitrary functions of ϕ andare related through equation (28), after setting the constant of integration to zero without any loss of generality. However, for canonical quantization, arbitrariness should be removed, since one is required to order the operators. In (27) we therefore removed the arbitrariness on β and γ following a simple assumption viz.

$$\beta_1 = \frac{\alpha_0}{12\lambda^2\phi} = \frac{\beta_{01}}{\phi}; \; \beta_2 = \frac{2\alpha_0}{\lambda^2\phi} = \frac{\beta_{02}}{\phi}; \; \gamma = -\frac{\phi^2}{48\lambda^2} = \gamma_0\phi^2$$

where $\beta_{01} = \frac{\alpha_0}{12\lambda^2}$; $\beta_{02} = \frac{2\alpha_0}{\lambda^2}$; $\gamma_0 = -\frac{1}{48\lambda^2}$ ...(29)

where, β_{01} and β_{02} are constants.

2.2.2 Inflation under Slow Roll Approximation

As before, we express (26) and (27) in terms of Hubble parameter for spatially at space ($k = 0$) as,

$$6\alpha H^2 = \frac{\dot{\phi}^2}{2} + [V - \left(6\alpha'\dot{\phi}H + 144\beta_1'\dot{\phi}H^3 - 12\beta_2'\dot{\phi}H^3 + 24\gamma'\dot{\phi}H^3)\right] \quad \ldots(30)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + [V' - \left(12\alpha'H^2 + 144\beta_1'\dot{\phi}H^4 - 12\beta_2'H^4 + 24\gamma'H^4\right] = 0 \quad \ldots(31)$$

respectively. Here again, instead of using additional hierarchy of flow parameters[22-25], we define a potential in the following manner:

$$U = V - 12H^2(\alpha + 12H^2\beta_1 - H^2\beta_2 + 2H^2\gamma) \qquad ...(32)$$

so that equation (31) takes the following standard form of Klein-Gordon Equation,

$$\ddot{\phi} + 3H\dot{\phi} + U' = 0 \qquad ...(33)$$

Clearly as before, the evolution of the scalar field is driven by the re-defined potential gradient $U' = \frac{dU}{d\phi}$, subject to damping by the Hubble expansion $3H\dot{\phi}$, as in the case of single field equation. Further, the potential $U(\phi)$carries all the information in connection with the coupling parameters of generalized higher order action under consideration. Further assuming,

$$U = V - 6H\dot{\phi}(\alpha' + 24H^2\beta_1' - 2H^2\beta_2' + 4H^2\gamma') \qquad ...(34)$$

equation (30) may be reduced to the following simplified form, viz.

$$6\alpha H^2 = \frac{\dot{\phi}^2}{2} + U(\phi) \qquad ...(35)$$

which is simply the Friedmann equation with a single scalar field and non-minimal coupling $\alpha(\phi)$. Here again,the two choices on the redefined potential $U(\phi)$made in (32) and (34), do not confront in any case, since the combination simply gives the evolution equation of the scalar field. During slow roll, the Hubble parameter Halmost remains unaltered. Thus replacing H by λ, and using the forms of the parameters $\alpha(\phi)$ presented in (28),along with $\beta_1(\phi)$, $\beta_2(\phi)$ and $\gamma(\phi)$ assumed in (29), the two relations (32) and (34) lead to the following first order differential equation on ϕ,

$$\left(\frac{\phi^3 - 6\alpha_0}{\phi^4}\right) d\phi = \frac{\lambda}{2} dt \qquad \text{... (36)}$$

which can immediately be integrated to yield,

$$\ln \phi + \frac{2\alpha_0}{\phi^3} = \frac{\lambda}{2(t - t_0)} \qquad \text{... (37)}$$

Clearly, if ϕ is not too large, ln ϕ remains subdominant, and ϕ falls-of with time, as expected during inflationary regime.

Now, under slow roll approximation $(\dot{\phi}^2 \ll U(\phi) \text{and}\, \ddot{\phi} \ll 3H|\dot{\phi}|)$, the effective Friedmann (35) and the Klein-Gordon (33) equations take the same form of equations (16) and (17). Hence $\frac{H}{\dot{\phi}} = -\frac{U}{2\alpha U'}$,as before. Therefore,the slow roll parameters and the number of e-folds, at which the present Hubble scale equals the Hubble scale during inflation, may be computed as:

$$\epsilon = -\frac{\dot{H}}{H^2} = \alpha \left(\frac{U'}{U}\right)^2 - \alpha' \left(\frac{U'}{U}\right); \;\; \eta = 2\alpha \left(\frac{U''}{U}\right). \qquad \text{... (38)}$$

$$N(\phi) \simeq \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi \simeq \int_{\phi_f}^{\phi_i} \left(\frac{U}{2\alpha U'}\right) d\phi \qquad \text{... (39)}$$

where, ϕ_i and ϕ_f denote the values of the scalar field at the beginning (t_i) and the end (t_f) of inflation. Thus, slow roll parameters reflect all the interactions, as exhibited earlier[23–25], via the redefined potential $U(\phi)$. Now, let us make the following choice of the redefined potential,

$$U = \frac{1}{2} m^2 \phi^2 - u_0, \text{where}, m^2 = \lambda^2 + H^2 \approx 2\lambda^2 \qquad \text{... (40)}$$

where u_0 is a constant which is essentially the vacuum energy density, i.e. the cosmological constant, that we omitted from the action. Thus, the slow roll parameters ϵ and η (38) and the number of e-folding N (39) takethe following forms,

$$\epsilon = \frac{4m^4\alpha_0\phi}{(m^2\phi^2 - 2u_0)^2} + \frac{2m^2\alpha_0}{(m^2\phi^3 - 2u_0\phi)}, \eta = \frac{4m^2\alpha_0}{m^2\phi^3 - 2u_0\phi}, \quad \ldots(41)$$

$$N = \frac{1}{4\alpha_0}\int_{\phi_f}^{\phi_i} \frac{(m^2\phi^2 - 2u_0)}{m^2} d\phi$$

$$= \frac{1}{12\alpha_0}\left(\phi_i^3 - \phi_f^3\right) - \frac{u_0}{2m^2\alpha_0}\left(\phi_i - \phi_f\right) \quad \ldots(42)$$

Here again we present two sets of data in table 3 and table 4 taking $\phi_i = 1.54M_P$; $m^2 = 0.9M_P^2$; $u_0 = 1M_P^4$, so that inflation ends ($\epsilon = 1$) around $\phi_f \approx 1.49M_P$. In table 3, we have varied u_0 in such a manner ($1.80 \times 10^{-5}M_P^3 < \alpha_0 < 2.15 \times 10^{-5}M_P^3$) that the scalar tilt, i.e. the spectral index lies very much within the specified range, i.e. 0.964 $<n_s<$ 0.970. The number of e-folds $42 \leq N \leq 50$ is enough to solve the horizon and the flatness problem. However, the tensor to scalar ratio does not admit value $r < 0.06$. On the contrary, in table 4, we havekept the tensor to scalar ratio within the specified limit $r < 0.06$, and find that the spectral index goes beyond experimental limit. Further, the number of e-folds becomes a bit large. Since, Planck's data has been analyzed following a particular model, so some deviation is expected for more involved models, under present consideration. In this case also it is important to mention that, the energy scale of inflation has been found to be sub-Planckian $(H_*) \approx 10^{-5}M_P$(27). The model also admits graceful exit from inflation, since the scalar field starts oscillating$\left(\phi \sim e^{i\sqrt{2}\lambda t}\right)$ many times over a Hubble time, driving a matter-dominated era at the end of inflation[27] .

Table–3

α_0 in $10^{-5} M_P^3$	ϕ_f in M_P	n_s	r	N
1.80	1.49419	0.9699	0.08201	50
1.85	1.49424	0.9690	0.08429	49
1.90	1.49429	0.9681	0.08657	48
1.95	1.49433	0.9674	0.08884	47
2.00	1.49438	0.9665	0.09112	46
2.05	1.49442	0.9657	0.09340	44
2.10	1.49447	0.9649	0.09568	43
2.15	1.49451	0.9640	0.09796	42

Data set for the inflationary parameters taking $\phi_i = 1.54M_P$; $m^2 = 0.9M_P^2$; $u_0 = 1M_P^4$ and varying α_0, keeping n_s within Planck's constraint limit.

Table–4

α_0 in $10^{-5} M_P^3$	ϕ_f in M_P	n_s	r	N
1.20	1.49355	0.9799	0.05467	76
1.22	1.49358	0.9796	0.05558	75
1.24	1.49360	0.9792	0.05650	73
1.26	1.49362	0.9789	0.05740	72
1.28	1.49365	0.9786	0.05832	71
1.30	1.49367	0.9782	0.05923	70

Data set for the inflationary parameters taking $\phi_i = 1.54M_P$; $m^2 = 0.9M_P^2$; $u_0 = 1M_P^4$ and varying α_0, keeping r within Planck's constraint limit.

2.3 Case-3

Although, Gauss-Bonnet term is constructed from higher order curvature invariant terms, the beauty lies in the fact that, it does not contain anything above second derivative, and hence is free from ghost degrees of freedom and also renormalizable. The problem is, it suffers from the pathology of `Branched Hamiltonian'[27, 28]. The presence of cubic kinetic term and quadratic constraints appearing through Gauss-Bonnet combination, makes the theory intrinsically nonlinear. Even its linearized version is cubic rather than quadratic. Since, the expression for velocities are multi-valued functions of momentum, it results in the so called multiply branched Hamiltonian with cusps. This makes classical solution unpredictable, as at any instant of time, one can jump from one branch of the Hamiltonian to the other. Further, the momentum does not provide a complete set of commuting observable, resulting in non-unitary time evolution of quantum states. Such a pronounced exotic behaviour does not allow Hamiltonian formulation following conventional Legendre transformation. There is no unique resolution to this issue. However, in (28) and (29), it was shown that the pathology may be bypassed by adding curvature squared term. Let us therefore take into account the action as considered earlier in (29) which is,

$$A = \int \left[\frac{R}{16\pi G} + \xi(\phi)(\mathcal{G} + \beta R^2) - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right]\sqrt{-g}\,d^4x \quad \dots (43)$$

Note that we have omitted scalar coupling with Einstein-Hilbert sector and introduced the same coupling parameter $\xi(\phi)$ with the Gauss-Bonnet and the R^2 term.

2.3.1 Field equations and classical solutions

The $\binom{0}{0}$ and the ϕ variation equations are,

$$\frac{\dot a^2}{a^2} = -96\beta\xi\pi G\left[\frac{2\dot a\dddot a}{a^2} - \frac{\ddot a^2}{a^2} + \frac{2\dot a^2\ddot a}{a^3} - \frac{3\dot a^4}{a^4}\right] - 192\beta\pi G\xi'\dot\phi\left(\frac{\dot a\ddot a}{a^2} + \frac{\dot a^3}{a^3}\right) - 64\pi G\xi'\dot\phi\left(\frac{\dot a^3}{a^3}\right) + \frac{8\pi G}{3}\left(\frac{\dot\phi^2}{2} + V\right) \quad \dots (44)$$

$$-24\xi'\dot a^2\ddot a - 36\beta\xi' a\ddot a^2 - 72\xi'\dot a^2\ddot a - 36\beta\xi'\frac{\dot a^4}{a} + 3a^2\dot a\dot\phi + a^3(\ddot\phi + V') = 0 \quad \dots (45)$$

If we now seek classical de-Sitter solution in the form

$$a = \alpha_0 e^{\lambda t} \text{and } \phi = \phi_0 e^{-\lambda t} \quad \dots (46)$$

then the coupling parameter and the potential are fixed as

$$\xi(\phi) = \xi\phi^{-2}, \text{ and } V(\phi) = V_1 + V_0\phi^2, \quad \dots (47)$$

restricting the constants to $V_1 = \frac{3\lambda^2}{8\pi G}$, $\beta = -\frac{1}{6}$, and $V_0 = -\frac{\lambda^2}{2}$, where ξ and λ are constants.

2.3.2 Inflation under Slow Roll approximation

Note that during inflation the Hubble parameter varies slowly and hence we can replace the constant λ by H,without loss of generality. Here, instead of redefining the potential as in the previous two cases, we consider an additional slow roll parameter viz. $\delta_1 = 4H\dot{\xi} \ll 1$, following the hierarchy of flow parameters[22-25.] Thus we have three slow roll parameters ϵ, η, δ_1 at hand, in view of which (44) and (45) may be approximated to (29),

$$H^2 \simeq \frac{1}{3M_P^2}V, \text{ and } H\dot{\phi} \simeq -\frac{1}{3}VQ, \quad \dots (48)$$

where $Q = \frac{V'}{V}$. Now, in view of the above form of a monomial potential and an inverse monomial GB coupling(47), namely $V(\phi) = V_1 + V_0\phi^2$ and $\xi(\phi) = \xi\phi^{-2}$, where V_0, V_1,ξ are constants, the slow roll parameters andthe number of e-folds may be expressed as,

$$\epsilon = \frac{2\phi^2 M_P^2}{\left(\frac{V_1}{V_0} + \phi^2\right)^2}, \qquad \eta = \frac{2V_0}{\left(\frac{V_1}{V_0} + \phi^2\right)}$$

$$N = \frac{1}{2M_P^2}\int_{\phi_f}^{\phi_i}\frac{(V_1 + V_0\phi^2)}{V_0\phi}d\phi$$

$$= \frac{1}{2M_P^2}\left[\frac{V_1}{V_0}\ln\left(\frac{\phi_i}{\phi_f}\right) + \frac{(\phi_i^2 - \phi_f^2)}{2}\right] \quad \ldots (49)$$

Table–5

$\frac{V_1}{V_0}$ in M_P^2	ϕ_f in M_P	r	n_s	N
-6.0	3.2566	0.1244	.9685	60
-5.5	3.1566	0.1240	.9687	60
-5.0	3.0523	0.1235	.9688	61
-4.5	2.9432	0.1231	.9690	61
-4.0	2.8284	0.1226	.9691	62
-3.5	2.7071	0.1221	.9693	62
-3.0	2.5780	0.1217	.9694	63
-2.5	2.4392	0.1212	.9696	64
-2.0	2.2883	0.1208	.9697	64
-1.5	2.1213	0.1203	.9698	65
-1.0	1.9319	0.1199	.9699	65

Data set for the inflationary parameters taking$\phi_i = 16.4M_P,\ V_0 = 1M_P^2$

In table 5 we present a data set under the choice $\phi_i = 16.4M_P$ and $V_0 = 1M_P^2$, , while $\frac{V_1}{V_0}$ is varied within therange $-6.0M_P^2 < \frac{V_1}{V_0} < -1.0\ M_P^2$. Although the spectral index of scalar perturbation lies within the experimental limit ($0.96 \leq n_s \leq\ 0.97$) and the number of e-folding ranges within $60\ <\ N\ <\ 65$, which is sufficient to solve the horizon and flatness problems, it has not been possible to keep the tensor to scalar ratio ($r <\ 0.1$) within the observational limit. Further, even though the model allows graceful exit from inflation, since the scalar field exhibit oscillatory behaviour as $\left(\phi\sim\sqrt{\frac{V_1}{V_2}}\sin\left(\sqrt{2t}\sqrt{V_0}\pm c_1\sqrt{V_0}\right)\right)$ at the end of inflation; however the energy scale of inflation is super Planckian $H_* \approx 9.38MP$. Nonetheless, before discarding this model, we need to apply redefined potential technique, instead of the additional slow roll parameterδ_1 , which we pose in the future.

2.4 Case-4

In this subsection, we shall consider yet another higher-order modified gravitational action, which has not been treated earlier. Gauss-Bonnet term being topologically invariant, does not contribute to the field equation, as already mentioned. Therefore, a dilatonic (scalar) coupling is necessary. Recently, $F(\mathcal{G})$ theory has been proposed as an alternative to the dark energy. It is interesting to note that different powers (other than one) of the Gauss-Bonnet term $\mathcal{G}$, may be incorporated in the action without dilatonic coupling. It has been found that a typicalform of $F(\mathcal{G}) = F_0\mathcal{G}^m + F_1\mathcal{G}^n$, might unify early inflation with the late-time cosmic acceleration[29] . Particularly for $m < \frac{1}{2}$,late-time acceleration may be addressed, while for $n > 1$, early inflation is admissible. Since we are interested in the evolution of the early universe, so we leave the first term and choose $n = 2$, for simplicity, and express the action in the presence of a Gauss-Bonnet dilatonic term, which as mentioned is an outcome of weak field approximation of different versions of string theory, as:

$$A = \int \left[\alpha(\phi)(R-2\Lambda) + \beta(\phi)\mathcal{G} + \gamma\mathcal{G}^2 - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right]\sqrt{-g}\,d^4x \quad \ldots(50)$$

2.4.1 Field equations and classical solutions

The $\binom{0}{0}$ and the ϕ variation equations in connection with the above action (50) in the background of Robertson-Walker metric (3) are the following:

$$2\alpha\left(\frac{3\dot{a}^2}{a^2}-\Lambda\right) + 18\gamma\left[64\left(\frac{3\dot{a}^6\ddot{a}}{a^7}+\frac{\dot{a}^5\dddot{a}}{a^6}\right)+96\left(\frac{\dot{a}^4}{a^4}+\frac{\dot{a}^2\ddot{a}}{a^3}\right) - 576\left(\frac{\dot{a}^8}{a^8}+\frac{\dot{a}^6\ddot{a}}{a^7}\right)+\frac{480\dot{a}^8}{a^8}\right]+\frac{6\alpha'\dot{\phi}\dot{a}}{a}+\frac{24\beta'\dot{\phi}\,\dot{a}^3}{a^3} = \frac{1}{2}\dot{\phi}^2+V(\phi) \quad \dots(51)$$

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}+V'-6\alpha'\left(\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}\right)+2\Lambda\alpha'-24\beta'\left(\frac{\dot{a}^4}{a^4}+\frac{\dot{a}^2\ddot{a}}{a^3}\right) + \frac{24\beta'\dot{a}^4}{a^4}=0 \quad \dots(52)$$

Now seeking inflationary solution of the above classical field equations in the following standard de-Sitter form,

$$a = a_0 e^{\lambda t};\ \phi = \phi_0 e^{-\lambda t} \quad \dots(53)$$

the parameters $\alpha,\ \beta$ and potential $V\ (\phi)$ are fixed as,

$$\alpha(\phi)=\frac{a_0}{\phi};\ \ \beta(\phi)=-\frac{\phi^2}{48\lambda^2}-\frac{\alpha_0}{\phi}\left(\frac{1}{2\lambda^2}-\frac{\Lambda}{12\lambda^4}\right)=-\frac{\alpha_0\beta_0}{\phi}-\beta_1\phi^2$$

$$V(\phi)=\frac{1}{2}\lambda^2\phi^2-576\gamma\lambda^8;\ \ \beta_0=\left(\frac{1}{2\lambda^2}-\frac{\Lambda}{12\lambda^4}\right);\ \beta_1=\frac{1}{48\lambda^2};\ \gamma=\gamma_0 \quad \dots(54)$$

where, $a_0,\ \alpha_0, \gamma_0, \phi_0,$ and λ are arbitrary constants, while constants $\beta_0,\ \beta_1$ are related though λ.

2.4.2 Inflation under Slow Roll Approximation

As before, let us express equations (51) and (52) in terms of the Hubble parameter H in the spatially at space($k\ =\ 0$) respectively as,

$$6\alpha H^2 = 576\gamma_0 H^8 - 6\alpha'\dot{\phi}H - 24\beta'\dot{\phi}H^3 + (V + 2\Lambda\alpha) + \frac{\dot{\phi}^2}{2} \qquad ...(55)$$

and

$$\ddot{\phi} + 3H\dot{\phi} = -V' - 2\alpha'\Lambda + 12\alpha' H^2 + 24\beta' H^4. \qquad ... (56)$$

As in case-1 and case-2, here again we reduce the above set of highly complicated equations by redefining the potential, instead of using the additional hierarchy of flow parameters[22-25]. For example, choosing the potential as,

$$U = V + 2\alpha\Lambda - 12H^2(\alpha + 2H^2\beta) \qquad ... (57)$$

the above consideration again modifies the equation (56) to the standard form of Klein-Gordon Equation as,

$$\ddot{\phi} + 3H\dot{\phi} + U' = 0 \qquad ... (58)$$

Further, assuming

$$U = V + 2\alpha\Lambda + 576\gamma_0 H^8 - 6H\dot{\phi}(\alpha' + 4H^2\beta') \qquad ...(59)$$

equation (55) may also be reduced to the following simplified form, viz.

$$6\alpha H^2 = \frac{\dot{\phi}^2}{2} + U(\phi) \qquad ... (60)$$

Here again we mention that, the two choices of the redefined potential $U(\phi)$ made in (57) and (59), do notcontradict each other, rather equating the two redefined potential, one obtains the following first order differential equation on ϕ (since the Hubble parameter being slowly varying, may be treated almost as a constant):

$$\dot{\phi}(t) = \frac{2H\alpha + 2H^3\beta + 96\gamma_0 H^7}{\alpha' + 4H^2\beta'} \qquad ... (61)$$

Shortly, the behaviour of ϕ with time will be exhibited. We now enforce the standard slow-roll conditions $\dot{\phi}^2 \ll U$ and $|\ddot{\phi}| \ll 3H|\dot{\phi}|$, on equations (60) and (58), which thus finally reduce to,

$$6\alpha H^2 \simeq U \qquad ...(62)$$

and

$$3H\dot{\phi} \simeq -U' \qquad \text{... (63)}$$

respectively. Let us now compute the functional form of $U = U(\phi)$. For this purpose, we consider the same quadratic form of the potential as, $V(\phi) = \frac{1}{2}\lambda^2\phi^2 - 576\gamma_0\lambda^8$, along with given forms of $\alpha(\phi)$, $\beta(\phi)$ in (54), which satisfy classical de-Sitter solutions. As already mentioned, during inflation the Hubble parameter remains almost constant, and therefore while computing $U(\phi)$, one can replace it by the constant $H \approx \lambda$, without any loss of generality. Thus from (57) one obtains,

$$U = \lambda^2\phi^2 - 576\gamma_0\lambda^8 = m^2\phi^2 - C_0 \qquad \text{...(64)}$$

where m may be treated as the mass of the scalar field and $C_0 = 576\gamma_0\lambda^8$. Now, for the above form of $U(\phi)$ (64), the slow roll parameters read as,

$$\epsilon = \frac{4m^4\alpha_0\phi}{(m^2\phi^2 - C_0)^2} + \frac{2m^2\alpha_0}{(m^2\phi^3 - \phi C_0)}, \qquad \eta = \frac{2m^2\alpha_0}{m^2\phi^3 - \phi C_0},$$

$$N = \frac{1}{4\alpha_0}\int_{\phi_f}^{\phi_i}\frac{(m^2\phi^2 - C_0)}{m^2}d\phi$$

$$= \frac{1}{12\alpha_0}\left(\phi_i^3 - \phi_f^3\right) - \frac{C_0}{8m^2\alpha_0}\left(\phi_i - \phi_f\right), \qquad \text{... (65)}$$

Here again, we compute the inflationary parameters taking $\phi_i = 3.40M_P$ and $m^2 = 1 \times 10^{-10}M_P^2$. In table 6 we consider $C_0 = 7.10 \times 10^{-10}M_P^4$ and vary α_0 within the range $0.0070M_P^3 < \alpha_0 < 0.0076M_P^3$, so that the spectral index lies within the experimental limit $0.966 < n_s < 0.970$. Although the number of e-folds remainwithin the range $51 < N < 56$, which is sufficient to solve the horizon and flatness problems, the tensor to scalar ratio cannot be reduced below $r = 0.09$. Therefore, although r fits fairly well with the Planck's data[10,11], it does not fall within the range specified by other experiments, viz. BAO, BICEP, BK15 Keck Array data. In table 7, we fix $C_0 = 6.35 \times 10^{-10} M_P^4$ and vary α_0 within the range $0.0075 M_P^3 < \alpha_0 < 0.0081 M_P^3$. As a result, the specified range of spectral index is relaxed. None the less, it is still not possible to keep $r < 0.07$.

Table–6

α_0 in M_P^3	ϕ_f in M_P	n_s	r	N
0.0076	2.61173	0.9668	0.0992	51
0.0075	2.61207	0.9673	0.0979	52
0.0074	2.61242	0.9677	0.0966	53
0.0073	2.61272	0.9681	0.0953	54
0.0072	2.61312	0.9686	0.0940	54
0.0071	2.61348	0.9690	0.0927	55
0.0070	2.61383	0.9694	0.0914	56

Data set for the inflationary parameters taking $\phi_i = 3.4 M_P; m^2 = 1 \times 10^{-10} M_P^2$; $C_0 = 7.10 \times 10^{-10} M_P^4$ and varying α_0, keeping n_s within Planck's constraint limit.

Table–7

α_0 in M_P^3	ϕ_f in M_P	n_s	r	N
0.0081	2.46388	0.9738	0.0796	67
0.0080	2.46423	0.9741	0.0786	68
0.0079	2.46457	0.9745	0.0776	69
0.0078	2.46492	0.9748	0.0766	69
0.0077	2.46527	0.9751	0.0756	70
0.0076	2.46562	0.9754	0.0747	71
0.0075	2.46598	0.9754	0.0737	72

Data set for the inflationary parameters taking $\phi_i = 3.40 M_P$; $m^2 = 1 \times 10^{-10} M_P^2$; $C_0 = 6.35 \times 10^{-10} M_P^4$ and varying α_0 in such a manner that r is small.

Let us therefore proceed to find the energy scale of inflation. In view of the above form of $U(\phi)$ (64), we obtain the following expression from equation(60),

$$6\frac{\alpha_0}{\phi}H^2 = m^2\phi^2 - C_0 \qquad ...(66)$$

Now, if we choose the value of $C_0 = 7.10 \times 10^{-10}M_P^4$, together with a value of $\alpha_0 = 0.0075M_P^3$ and take $m^2 = 1 \times 10^{-10}M_P^2, \phi_i = 3.40M_P$, as depicted in the table-6, we simply find,

$$H^2 = \frac{(m^2\phi^3 - C_0\phi)}{6\alpha_0}, \text{andhence,} \qquad H^2 \approx 3.37 \times 10^{-8}M_P^2 \qquad ...(67)$$

Therefore, the energy scale of inflation has been found to be sub-Planckian ($H_* \approx 10^{-4}M_P$).

To exhibit consistency of our choice of redefined potential presented in (54), we treat the Hubble parameter to be nearly constant during inflation $H \approx \lambda$. As a result, equation (61) may now be expressed as,

$$\dot{\phi}(t) = \frac{\lambda\phi[2C_0\phi + 4\Lambda\alpha_0 - \lambda^2\phi^3]}{4\alpha_0(3\lambda^2 - \Lambda) - 2\lambda^2\phi^3} \qquad ...(68)$$

The above differential equation cannot be integrated analytically. None the less, since $C_0 \approx 10^{-10}M_P^4$ and $\lambda^2 \approx 10^{-8}M_P^2$, the terms associated with these parameters may be neglected from the numerator and denominator. Asa result, (68) may be suitably approximated to,

$$\dot{\phi}(t) = -\left[\frac{\lambda\phi(4\alpha_0\Lambda)}{4\alpha_0\Lambda}\right], \qquad \text{that is } \dot{\phi} = -\lambda\phi \qquad ...(69)$$

This is a remarkable outcome, since even after making an additional assumption of redefined potential, the scalar field evolves identically as in the classical de-Sitter solution (53). Having shown that ϕ decays, let us now express equation (60) as,

$$\frac{3H^2}{m^2} = \frac{\phi}{2\alpha_0}\left(\frac{\dot{\phi}^2}{2m^2} + \phi^2 - \frac{C_0}{m^2}\right) \qquad ...(70)$$

Note that for single scalar field, the above equation reads as: $3H^2 = \frac{1}{2M_P^2}\left(\dot{\phi}^2 + 2m^2\phi^2 - 2C_0\right)$. Since at the end of inflation, $\frac{\phi}{2\alpha_0} \sim 226 M_P^{-2}$, according to the present data set, so once the Hubble rate (*H*) falls below m, this equation (70) may be approximated to,

$$\dot{\phi}^2 \approx -2(m^2\phi^2 - C_0), \quad \ldots (71)$$

which may immediately be integrated to yield,

$$\phi(t) = \pm \frac{\sqrt{C_0}\tan\left[m\left(\sqrt{2}t - t_0\right)\right]}{m\sqrt{\tan^2\left[m\left(\sqrt{2}t - t_0\right)\right] + 1}} \quad \ldots (72)$$

and may further be simplified to obtain

$$\phi(t) = \pm \frac{\sqrt{C_0}}{m}\sin\left[m\left(\sqrt{2}t - t_0\right)\right] \quad \ldots (73)$$

Where t_0 is the constant of integration. Thus the scalar field starts oscillating many times over a Hubble time, driving a matter-dominated era as inflation ends.

2.5 Case-5

All the modified theories of gravity considered so far are higher order theories. There is yet another class of modified theories, viz. the non-minimally coupled scalar tensor theories of gravity. Such theories are essentially dark energy quintessence models. Here we consider a pure (having regular kinetic energy term and being devoid of higher-order terms) non-minimally coupled scalar-tensor theory of gravity, as considered in earlier[30] ,for which the action is expressed in the form,

$$A = \int \left[f(\phi)R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} - V(\phi)\mathcal{L}_m\right]\sqrt{-g}\,d^4x \quad \ldots (74)$$

where, $\mathcal{L}_m$ is the matter Lagrangian density, $f(\phi)$ is the coupling parameter, while, $\omega(\phi)$ is the variable Brans-Dicke parameter.

2.5.1 Field equations and classical solutions

The general field equations corresponding to action (74) are,

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)f(\phi) + g_{\mu\nu}\Box f(\phi) - f_{;\mu;\nu} - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} + V(\phi)\right) = T_{\mu\nu} \quad ...(75)$$

$$Rf' + 2\frac{\omega(\phi)}{\phi}\phi + \left(\frac{\omega'(\phi)}{\phi} - \frac{\omega(\phi)}{\phi^2}\right)\phi_{,\mu}\phi^{,\mu} - V'(\phi) = 0 \quad ...(76)$$

where prime denotes derivative with respect to ϕ, and $\Box$ denotes D'Alembertian, such that, $\Box f(\phi) = f''\phi_{,\mu}\phi^{,\mu} - f'\Box\phi$ The model involves three functional parameters viz. the coupling parameter $f(\phi)$, the Brans-Dicke parameter$\omega(\phi)$and the potential $V(\phi)$. It is customary to choose these parameters by hand in order to study the evolution of the universe. However, we have proposed a unique technique to relate the parameters in such a manner, that choosing one of these may fix the rest[31–34]. We have shown that there exists a general conserved current which is admissible by the above pair of field equations, as demonstrated in[30-35] leading to

$$V(\phi) \propto f(\phi)^2 \quad ...(77)$$

It is convenient and hence customary to study inflationary evolution in the Einstein's frame under suitable transformation of variables, where possible. In the very early vacuum dominated era, symmetry holds, and thus we can express the action (74) in the form,

$$A = \int \left[f(\phi)R - \frac{K(\phi)}{2}\phi_{,\mu}\phi^{,\mu} - V'(\phi)\right]\sqrt{-g}\,d^4x \quad ...(78)$$

where, $K(\phi) = 2\frac{\omega(\phi)}{\phi}$. Under a conformal transformation[36]

$$g_{E\mu\nu} = f(\phi)g_{\mu\nu}, \quad ...(79)$$

the above action (78) may be translated to the following Einstein's frame action as,

$$A = \int \left[R_E - \frac{1}{2}\sigma_{E,\mu}\sigma_E^{,\mu} - V_E\big(\sigma(\phi)\big)\right]\sqrt{-g_E}\,d^4x \qquad \dots (80)$$

where, the subscript `E' stands for Einstein's frame. The effective potential V_E and the transformed scalar fieldσin the Einstein's frame may be found from the following expressions,

$$V_E = \frac{V(\phi)}{f^2(\phi)};\ \text{and,}\ \left(\frac{d\sigma}{d\phi}\right)^2 = \frac{K(\phi)}{f(\phi)} + 3\frac{f'^2(\phi)}{f^2(\phi)}$$

$$= \frac{2\omega(\phi)}{\phi f(\phi)} + 3\frac{f'^2(\phi)}{f^2(\phi)} \qquad \dots (81)$$

2.5.2 Inflation under Slow Roll approximation

Inflation with such a non-minimally coupled scalar-tensor theory of gravity is undergoing serious investigation over several decades[37–47]. In view of the action (80), one can cast the field equations, viz. the Klein-Gordon and the $\binom{0}{0}$ equations of Einstein in the background of Robertson-Walker metric (3)as,

$$\ddot{\sigma} + 2H\dot{\sigma} + k_0 V_E' = 0, \qquad 3H^2 = \frac{1}{2}\dot{\sigma}^2 + k_0 V_E \qquad \dots (82)$$

where, the Hubble parameter is defined as $H = \frac{\dot{\alpha}_E}{\alpha_E}$, and $k_0 = 1M_P^2$, while the slow-roll parameters and the numberof e-folding take the following forms,

$$\epsilon = \left(\frac{V_E'}{V_E}\right)^2\left(\frac{d\sigma}{d\phi}\right)^{-2};\ \eta = 2\left[\left(\frac{V_E''}{V_E}\right)\left(\frac{d\sigma}{d\phi}\right)^{-2} - \left(\frac{V_E'}{V_E}\right)^2\left(\frac{d\sigma}{d\phi}\right)^{-3}\frac{d^2\sigma}{d\phi^2}\right];$$

$$N = \int_{t_i}^{t_f} H dt = \frac{1}{2}\int_{\phi_e}^{\phi_b} \frac{d\phi}{\sqrt{\epsilon}}\frac{d\sigma}{d\phi}. \qquad \dots (83)$$

In the above, t_i and t_f denote time for the beginning and the end of inflation respectively.

1. Quadratic Potential

We should choose the same form of quadratic potential for the comparative study under consideration. Thus, in view of the symmetry, (77) if we choose $f(\phi) = f_0\phi$ then $V(\phi) = m^2\phi^2 + C_0$, where we have added a constant C_0 in the potential without loss of generality. The parameters of the theory under consideration can therefore be expressed as (31),

$$\omega(\phi) = \frac{\omega_0^2 - 3f_0^2}{2f_0}, \qquad \frac{d\sigma}{d\phi} = \frac{\omega_0}{f_0\phi}, \qquad V_E = \frac{1}{f_0^2}(m^2 + C_0\phi^{-2}),$$

$$\epsilon = \frac{4f_0^2C_0^2}{\omega_0^2(C_0+m^2\phi^2)^2}, \quad \eta = \frac{8f_0^2C_0}{\omega_0^2(C_0+m^2\phi^2)},$$

$$N = \frac{\omega_0^2}{4f_0^2C_0^2}\left[m^2\left(\frac{\phi_i^2}{2} - \frac{\phi_f^2}{2}\right) + C_0(\ln\phi_i - \ln\phi_f)\right] \qquad \ldots(84)$$

In view of the above forms of the slow roll parameters (84), we present table-8, underneath, corresponding to $m^2 > 0$. The wonderful fit with the latest data sets released by Planck[10,11] is particularly significant because, $0.959 < n_s < 0.970$, while $r < 0.033$. Further, the number of e-fold ($40 \leq N \leq 53$) is sufficient to alleviate the horizon and flatness problems.

Now using the above form of V_E (81) and taking the values of m^2, C_0, ϕ_i from table-8, we obtain the followingexpression from equation (82),

$$3H^2 = k_0V_E = \frac{1}{f_0^2}\left(m^2 + \frac{C_0}{\phi^2}\right)M_P^2 \approx 10 \times 10^{-14}M_P^2 \qquad \ldots(85)$$

Hence, the energy scale of inflation has been found to be sub-Planckian $H_* \approx 10^{-7}M_P$. Additionally, this model also gracefully exits from inflationary regime, since the scalar field exhibits oscillatory behaviour $\phi \sim \frac{1}{2m^2}\left[(1-m^2C_0)\cos\left(\frac{\sqrt{2m^2}}{\omega_0}t\right)\right]$, at the end of inflation. Thus the scalar field starts oscillating many times over a Hubble time, driving a matter-dominated era at the end of inflation.

Table–8

ϕ_f in M_P	ω_0 in M_P	$r = 16\epsilon$	n_s	N
1.01905	6.5	0.03192	.9605	41
1.01802	6.6	0.03096	.9617	42
1.01702	6.7	0.03004	.9629	44
1.01605	6.8	0.02917	.9639	45
1.01510	6.9	0.02833	.9650	46
1.01419	7.0	0.02752	.9660	48
1.01329	7.1	0.02675	.9669	49
1.01242	7.2	0.02601	.9678	50
1.0116	7.3	0.02531	.9687	52
1.0107	7.4	0.02463	.9696	53
1.0099	7.5	0.02397	.9704	55

Data set for the inflationary parameters taking, $f(\phi) = f_0\phi, f_0 = \frac{1}{2}M_P, \phi_i = 2.0M_P$, $C_0 = -0.9\times10^{-13}M_P^4, m^2 = 1.0\times10^{-13}M_P^2$.

Such astounding fit with the experimental data provokes to study the late-stage of cosmic evolution. In this connection, we mention that the choice of the quadratic form of potential was undertaken due to the fact that de-Sitter solution for all the four higher-order modified theories of gravity considered here, restricts the potential in its quadratic form only. Nonetheless, in an early work[35], a quartic potential was taken into account, and it was shown that it can account for the late-stage of cosmic evolution, with excellence. In the following we therefore study inflation in view of quartic potential.

2. Quartic potential

In an earlier work[35], relaxing the symmetry, the coupling parameter and the potential were chosen as $f(\phi) = \phi^2$, and $V(\phi) = V_0\phi^4 - B\phi^2$, so that, the parameters of the theory under consideration can be expressed as,

$$\omega(\phi) = \frac{\omega_0^2 - 12\phi^2}{2\phi}, \quad \frac{d\sigma}{d\phi} = \frac{\omega_0}{\phi^2}, \; V_E = V_0 - \frac{B}{\phi^2}, \qquad \dots (86)$$

$$\epsilon = \frac{4B^2\phi^2}{\omega_0^2(V_0\phi^2 - B)^2}, \qquad \eta = -\frac{4B\phi^2}{\omega_0^2(V_0\phi^2 - B)},$$

$$N = \frac{\omega_0^2}{4B}\left[V_0 \ln(\phi_i - \phi_f) + \frac{B}{2}\left(\frac{1}{\phi_i^2} - \frac{1}{\phi_f^2}\right)\right] \qquad \dots (87)$$

It is important to mention that the same form of potential was also considered to study late-time cosmic acceleration[48]. The reason for such a choice of the potential was also clarified in[35]. In a nut-shell: in the non-minimal theory, the flat section of the potential $V(\phi)$, responsible for slow-rollover, is usually distorted. However, generalizing the form of non-minimal coupling by an arbitrary function $f(\phi)$, Park and Yamaguchi[49] could show that the flat potential required for slow roll, is still obtainable when V_E is asymptotically constant. Here, initially when $\phi \gg \sqrt{\frac{B}{V_0}}$, the second term may be neglected, so that $V_E \approx V_0$, and the potential becomes flat, admitting slow roll. Inflationary parameters were found to fit experimental data released (Planck's collaboration-2016) by the time the work[35] was carried out. However, over years, Planck's data puts up tighter constraints on inflationary parameters, and so it is quite reasonable to check, if this form of the quartic potential passes the said constraintlimits[10, 11].

Case-A

In the following table 9, we present our computed results on inflationary parameters, taking $\phi_i = 1.3M_P$, $B = 1.0 \times 10^{-20} M_P^2$ and $V_0 = 1.1 \times 10^{-20}$. It is found that the spectral index of scalar perturbation rangesbetween $0.959 < n_s < 0.970$, the tensor to scalar ratio remains around $r \approx 0.06$, and the number of e-folding is around $N \approx 50$, showing marvellous fit with the currently released data set[10, 11].

Table – 9

ϕ_f in M_P	ω_0 in M_P	$r = 16\epsilon$	n_s	N
.9753	42	0.08300	.9599	38
.9748	43	0.07927	.9618	40
.9743	44	0.07571	.9636	42
.9739	45	0.07238	.9650	44
.9734	46	0.06927	.9666	46
.9730	47	0.06636	.9680	48
.9726	48	0.06362	.9693	50
.9722	49	0.06105	.9706	52

Data set for the inflationary parameters taking,$f(\phi) = \phi^2, \phi_i = 1.3M_P$, $B = 1.0 \times 10^{-20} M_P^2, V_0 = 1.1 \times 10^{-20}$.

Let us therefore proceed to find the energy scale of inflation. In view of the above form of V_E (86), and using(81), we obtain the following expression from equation (82),

$$3H^2 \approx k_0 V_E = \left(V_0 - \frac{B}{\phi^2}\right) M_P^2 \approx 50.8 \times 10^{-22} M_P^2 \qquad ...(88)$$

The numerical value of $3H^2$ is an outcome of the values of V_0, B, and ϕ_i presented in table-9. Clearly, the energy scale of inflation is sub-Planckian ($H_* \approx 4.11 \times 10^{-11} M_P$). Exact solution of equation (82) using Mathematica,does not evince oscillatory behaviour of the scalar field at the end of inflation. We therefore choose an oscillatory ϕ a-priori, and see if the consequence is physically admissible. Let us therefore assume,

$$\phi = \exp(i\omega t) \qquad \ldots(89)$$

Equation (82) may therefore be expressed as,

$$6H^2 = 2k_0\left(V_0 - \frac{B}{\phi^2}\right) - \frac{\omega^2\omega_0^2}{\phi^2} \qquad \ldots(90)$$

Now taking the numerical values of the parameters from table-9, viz. $V_0 = 1.1 \times 10^{-20},\ B = 1.0 \times 10^{-20} M_P^2,\ \omega_0 = 45M_P,$
$\phi_f = 0.973M_P,$ and $k_0 = 1M_P$, we get,

$$6H^2 \approx 8.58 \times 10^{-22} M_P^2 - 2139\omega^2 M_P^2 \qquad \ldots(91)$$

If $\omega^2 = 1 \times 10^{-25}$, then

$$6H^2 \approx 6.441 \times 10^{-22} M_P^2,\ \text{or},\ H^2 \approx 1.07 \times 10^{-22} M_P^2 \qquad \ldots(92)$$

Hence, $H = 1.03 \times 10^{-11} M_P$. As repeatedly mentioned, Hubble parameter remains almost constant during inflation. It is called the scale of inflation, which is, $H_* \approx 4.11 \times 10^{-11} M_P$ in the present model. As inflation halts ($\epsilon = 1$) Hubble parameter decreases fast and we observe that as it reaches one-fourth the value of theHubble scale ($H_* = 1.03 \times 10^{-11} M_P$), the scalar field starts oscillating many times over a Hubble time, driving a matter-dominated era at the end of inflation. Consequently, graceful exit from inflation is also evinced.

Case-B

Quartic potential under current consideration has magical enchantment. The reason is, one can change the parameters over a wide range, and yet end up with outstanding data fit. In what follows, we show that, even setting both the parameters B and V_0 to negative values, and there after also interchanging their values, amazingly nice fit with Planck's data is realized. In the table 10, we present our computed results on inflationary parameters, taking $\phi_i = 1.5M_P, B = -1.1 \times 10^{-20}M_P^2$ and $V_0 = -1.0 \times 10^{-20}$. The spectral index of scalar perturbation isfound to range between $0.960 < n_s < 0.970$, while the tensor to scalar ratio ranges between $0.059 < r < 0.078$, and the number of e-folding is around $N \approx 50$. In table 11, on the contrary, the values of B and V_0 are interchanged,and the data set remains almost unaltered. Clearly, the data sets in both situations exhibit magnificent fit with the currently released data set[10, 11].

Table–10

ϕ_f in M_P	ω_0 in M_P	$r = 16\epsilon$	n_s	N
1.07598	41	0.07838	.9604	38
1.07533	42	0.07468	.9622	41
1.07470	43	0.07125	.9640	43
1.07471	44	0.06805	.9656	45
1.07354	45	0.06506	.9671	47
1.07299	46	0.06226	.9685	49
1.07247	47	0.05964	.9698	51

Data set for the inflationary parameters taking, $f(\phi) = \phi^2$, $\phi_i = 1.5M_P, B = -1.1 \times 10^{-20}M_P^2, V_0 = -1.0 \times 10^{-20}$.

Table–11

ϕ_f in M_P	ω_0 in M_P	$r = 16\epsilon$	n_s	N
.9753	42	0.08309	.9599	38
.9748	43	0.07927	.9618	40
.9743	44	0.07571	.9635	42
.9739	45	0.07238	.9650	44
.9734	46	0.06927	.9666	46
.9730	47	0.06636	.9680	48
.9726	48	0.06362	.9693	50
.9722	49	0.06105	.9706	52

Data set for the inflationary parameters taking,$f(\phi) = \phi^2$, $\phi_i = 1.3M_P$, $B = -1.0 \times 10^{-20}M_P^2$, $V_0 = -1.1 \times 10^{-20}$.

The energy scale of inflation is sub-Planckian ($H_* \approx 10^{-11}M_P$) taking into account $k_0 = -1M_P^2$ for both thedata sets presented in table 10 and table 11. The oscillatory behaviour of the scalar field is as exhibited earlier.

The beauty of the quartic potential with non-minimally coupled scalar-tensor theory of gravity was explored earlier, in connection with the later stage of cosmic evolution, taking into account the thermodynamic pressure (p) and the energy density (ρ) of the baryons and the CDM[35]. Here, we brief the outcome. In the radiation dominated era $(p = \frac{1}{3}\rho)$, the scalar field admits a solution in the form, $\phi = \frac{\phi_0}{\sqrt{(At-t_0)}}$, while the scalefactor evolves like the usual Friedmann solution, viz. $a = a_0\sqrt{(At - t_0)}$. In the pressure-less dust dominated era, the scale factor admits a solution $(a = a_0 \sin ht^{\frac{2}{3}})$, which had been graphically illustrated in[35]. The graphical representation depicts that at the early stage of the pressure-less dust $(p = 0)$ era, the universe had undergone Friedmann-like decelerated expansion $a \propto t^{2/3}$, while accelerated expansion initiated at the late stage of cosmic evolution around red-shift $z \approx 0.78$, which is in perfect agreement with experimental data. Additionally, other cosmological parameters were computed and it was found that:

1. The present value of the scale factor is exactly $(a_0 = 1.0)$,
2. The present value of the Hubble parameter is $H_0 = 69.24\ Km.s^{-1}Mpc^{-1}$,
3. The age of the universe is $t_0 = 13.86\ Gyr$ and hence,
4. $H_0 t_0 = 1.01$,

5. The deceleration parameter q remains almost constant $q \approx 0.5$, till the value of redshift $z = 4.0$, confirming a long Friedmann-like matter dominated era,
6. The present value of deceleration parameter is $q = -0.59$,
7. The present value of the effective state parameter is therefore, $\omega_{eff0} = -0.7$,
8. Considering, as usual, that the CMBR temperature falls as a^{-1} , and its value at decouplingto be $T_{dec} \approx 3000K$, the present value of it has been found to be $T_0 = 2.7255K$.

All these agree perfectly with current observation.

3. Concluding remarks

Seeds of structure in the universe are the density variations known as the primordial fluctuations. The prevalent and most widely accepted theory that can explain the origin of the seeds of perturbation is the cosmic inflation, which occurred soon after Planck's epoch $\left(t_p = 10^{-43}s\right)$. According to inflationary paradigm, the exponential growth of the scale factor caused quantum fluctuation of the inflation field (the scalar field that we considered here) to be stretched beyond the horizon and freeze. Later, as inflation halts, these seeds of perturbation enter the horizon and form structures. Primordial fluctuations are typically described by a power spectrum, which gives the power of variation of the function of spatial scale. Both the scalar and the tensor fluctuations follow a power law. The ratio of tensor to the scalar power spectra, called the tensor to the scalar ratio, is given by $r = 2\frac{|\delta_h|^2}{|\delta_R|^2}$, where, $|\delta_h|^2$ and $|\delta_R|^2$ are the tensor and scalar modes of perturbation respectively, and the factor 2 arises due to the presence of two polarizations of tensor modes. While Planck's collaboration teams[10,11] alone constrain $r < 0.1$, the combined data from other experiments viz. BAO, BICEP2, and BK15 Keck Array, tightens the constraint to $r < 0.06$. Now, the scale-dependence of the CMB power spectrum constrains the slope of the primordial scalar power spectrum, conventionally parameterized by the power-law index n_s, where $n_s = 1$ corresponds to a scale-invariant spectrum. The matter and baryon densities also affect the scale-dependence of the CMB spectra in a way that differs from a variation in n_s, leading to relatively mild degeneracies between these parameters. Assuming that the primordial power spectrum is an exact power law, we find $n_s = 0.9649 \pm .0042$ which is 8σ away from scale-invariance ($n_s = 1$). Further, BAO data also tightens the $|n_s|$constraint by a little amount. Combining all data, viz. TT, TE, EE + low E + lensing + BK15 + BAO, r is constrained even further to $r < 0.058$ with $n_s = 0.9668 \pm 0.0037$.

It is therefore worth to check the viability of different gravitational actions proposed over years, in connection with the currently available inflationary data sets. Note that all the experimental data are analyzed with a standard model viz. the single minimally coupled scalar field model. Hence, the result is expected to vary slightly, depending on the models. In this sense, all the four higher order theories taken up in the present analysis show quite a nice fit with the experimental data sets. However, one can deselect case 3, since the energy scale of inflation is super-Planckian, although further investigation is necessary. It is also required to see if the other three models show Friedmann-like behaviour ($a \propto \sqrt{t}$) in the radiation dominated era which initiated soon after the graceful exit from inflation. It is further suggestive to check if these models exhibit a long Friedmann-like pressure-less dust dominated era $(a \propto t^{2/3})$, after photons decoupled and prior to the recent accelerating phase. Analytical solutions are not available for these complicated models, and future task is to numerically simulate these models in the matter dominated eras. On the contrary, the non-minimally coupled scalar-tensor theory (case 5) show excellent fit with the available data and passes the tightest constraints imposed on the inflationary parameters. Since de-Sitter solution for all the higher order theories under consideration is admissible with standard square law potential $(V = m^2\phi^2)$, we fixed it for the non-minimally coupled scalar-tensor theory of gravity too. It is therefore required to see if the model with square law potential, potentially behaves in the matter-dominated era also. Nevertheless, with a different (quartic) potential this has been achieved earlier, which showed excellent agreement with FLRW model until recently, before it enters an accelerated phase of expansion.

There is a recent claim for direct detection of dark energy[50]. XENON1T, operating thousands of feet underground the Italian mountain `Monte Gran Sasso', is the most sensitive detector on earth searching for WIMP (Weakly Interacting Massive Particle) dark matter. Last year, it reported 53 excess recoil electrons than estimated. This was a great puzzle. In a recent publication[50] the authors assumed interaction of dark energy with the electro-magnetic field and followed a method called chameleon screening for their analysis. They inspected the effect on the detector, if dark energy is produced in a particular region of the sun, called tachocline, where magnetic field is very strong. To their surprise they found the excess recoil electrons are the outcome of dark energy. Future experiments will be able to confirm the claim. In this sense, the non-minimally coupled scalar-tensor theory of gravity, is highly promising. However, one cannot avoid the presence of higher-order curvature invariant terms in the very early universe. It is therefore suggestive to supplement action (74) at least with R^2 term to test the outcome. This may be posed in the future.

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On Almost Generalized Special Pseudo RICCI Symmetric Manifolds

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[**Abstract**: The object of this paper is to study a new type of non-flat Riemannian manifold called almost generalized special pseudo Ricci symmetric manifold. An existence theorem is given. It is shown that such a manifold with zero scalar curvature is even dimentional and such a manifold with non-zero scalar curvature is a quasi Einstein manifold if it is conformally flat Ricci semi symmetric].

Keywords: *Pseudo Ricci symmetric manifold, almost generalized pseudo Ricci symmetric, Special pseudo Ricci symmetric manifold, Cyclic parallel Ricci tensor, Conformally flat manifold, Ricci semi symmetric manifold.*

1.Introduction

In a paper M. C. Chaki[1] introduced and studied a type of non-flat Riemannian manifold called pseudo Ricci symmetric manifold.

According to him, a non-flat Riemannian manifold (M^n,g), (n>2), is called pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X),$ …(1.1)

for every vector field X, Y, Z , where A is a non-zero 1-form defined by $g(X,\rho) = A(X), \nabla$ denotes the operator of covariant differentiation with respect to the metric tensor g. A is called associated 1-form and ρ is called associated vector field. An n-dimensional manifold of this kind was denoted by the symbol $(PRS)_n$.

Generalizing the notion of pseudo Ricci symmetric manifold in a paper P. Chakrabarti and the present author[2] studied a new type of non-flat Riemannian manifold called almost generalized pseudo Ricci symmetric manifold. A non-flat Riemannian manifold (M^n,g), $(n > 2)$, is called almost generalized pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$(\nabla_X S)(Y,Z) =$

$[A(X) + B(X)]S(Y,Z) + C(Y)S(Z,X) + D(Z)S(X,Y),$ ... (1.2)

for every vector field X, Y, Z , where A, B, C , D are four non-zero 1-forms , ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. An n-dimensional manifold of this kind was denoted by the symbol $(AGPRS)_n$. Many works [Chaki and Kawaguchi[3]], [Chaki and Koley[4]] etc. have been done on pseudo Ricci symmetric manifolds and its different type of generalizations by several authors.

In a recent paper the present author [Saha[5]] defined and studied a type of non-flat Riemannian manifold called special pseudo Ricci symmetric manifold. According to him, a non-flat Riemannian manifold (M^n,g), (n>2), is called special pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(LY,Z) + A(Y)S(LZ,X) + A(Z)S(LX,Y), \quad \dots (1.3)$$

for every vector field X, Y, Z , where A and ∇ have the meanings already mentioned and L is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S of type (0, 2) and it is defined by

$$g(LX,Y) = S(X,Y), \quad \dots (1.4)$$

for any vector field X, Y. An n-dimensional manifold of this kind was denoted by the symbol $(SPRS)_n$.

Subsequently, the present author[6,7] defined and studied two type of Riemannian manifolds called almost special pseudo Ricci symmetric manifold and generalized special pseudo Ricci symmetric manifold which were defined respectively by the following equations

$$(\nabla_X S)(Y,Z) = [A(X) + B(X)]S(LY,Z) + A(Y)S(LZ,X) + A(Z)S(LX,Y), \quad \dots (1.5)$$

$$(\nabla_X S)(Y,Z) = 2A(X)S(LY,Z) + B(Y)S(LZ,X) + C(Z)S(LX,Y), \quad \dots (1.6)$$

where A, B and C are non-zero 1-forms and S, L and ∇ have the meanings already mentioned.

The aim of this paper is to define and study a type of non-flat Riemannian manifold called almost generalized special pseudo Ricci symmetric manifold. A non-flat Riemannian manifold $(M^n, g), (n > 2)$, is called aimost generalized special pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X) + B(X)]S(LY,Z) + C(Y)S(LZ,X) + D(Z)S(LX,Y), \quad \text{... (1.7)}$$

for every vector field X, Y, Z , where A, B, C and D are four non-zero 1-forms defined by $g(X,\alpha) = A(X)$, ... (1.8)

$g(X,\beta) = B(X)$ ...(1.9)

$g(X,\gamma) = C(X)$, ...(1.10)

$g(X,\delta) = D(X)$, ...(1.11)

L and ∇ have the meanings already mentioned. A, B, C and D are called associated 1-forms of the manifold and α,β,γ and δ are called associated vector fields of the manifolds.. An n-dimensional manifold of this kind shall be denoted by the symbol $(AGSPRS)_n$. If A = B , then (1.7) reduces to (1.6) i.e. the manifold becomes generalized special pseudo Ricci symmetric manifold, so the name ' almost generalized special pseudo Ricci symmetric manifold has been chosen.

2. Preliminaries

In this section we shall obtain some formula which will be used in sequel. Let (M^n, g) be a Riemannian manifold and { e_i}, i = 1, 2, ……, n be an orthonormal basis of the tangent space at each point and i is summed for $1 \leq i \leq n$. Let r be the scalar curvature of the manifold defined by

$S(e_i, e_i) = r$. … (2.1)

Since S is a symmetric ,we have from (1.4)

$S(LX,Y) = S(X,LY)$. …(2.2)

Putting Y = Z = e_i in (1.7) and i is summed for $1 \leq i \leq$n we get

$dr(X) = [A(X) + B(X)]\,|S|^2 + S(LX,\gamma) + S(LX,\delta)$, …(2.3)

where $|S|$ is the length of the Ricci tensor defined by $|S| = \sqrt{S(Le_i, e_i)}$.

Again from (1.7) we have

$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = [A(X) + B(X) - D(X)]S(LY,Z) -$

$[A(Z) + B(Z) - D(Z)]S(LX,Y).$ …(2.4)

3. *Existence of* $(AGSPRS)_n$

In this section we shall give an existence theorem of the almost generalized special pseudo Ricci symmetric manifold. To show the existence we shall obtain a necessary and sufficient condition so that a $(AGPRS)_n$ can be a $(AGSPRS)_n$.

Let us suppose that a $(AGPRS)_n$ satisfies the condition

$L^2X = LX.$ …(3.1)

From (1.4) and (3.1) we get

S(LX, Y) = S(X Y). …(3.2)

From (1.2), (2.2) and (3.2) we get

$(\nabla_X S)(Y,Z) = [A(X) + B(X)]\, S(LY,Z) + C(Y)S(LZ,X) +$

$D(Z)S(LX,Y),$ … (3.3)

which shows that the manifold is a $(AGSPRS)_n$. Hence we can state the following theorem:

Theorem 3.1: A $(AGPRS)_n$ is a $(AGSPRS)_n$ with the same associated 1-forms if L^2X = LX.

This gives the existence of $(AGSPRS)_n$. Conversely, if $(AGSPRS)_n$ satisfies the condition (3.1), then from (2.2) and (1.7) we get

$(\nabla_X S)(Y,Z) = [\, A(X) + B(X)]S(Y,Z) + C(Y)S(X,Z) + D(Z)S(Y,X),$

… (3.4)

which shows that the manifold is a $(AGPRS)_n$. Hence we can state the following theorem:

Theorem 3.2: A $(AGSPRS)_n$ is a $(AGPRS)_n$ with the same associated 1-forms if L^2X = LX.

Now Theorem 3.1 and Theorem 3.2 lead us to the following theorem:

Theorem 3.3: A $(AGPRS)_n$ is a $(AGSPRS)_n$ with the same associated 1-forms if and only if both the manifolds satisfy the condition L^2X = LX.

Note: The present author has defined and studied a type of Riemannian manifold called Ricci Riemannian manifold[8,9] and it is defined as follows:

A non-flat Riemannian manifold (M^n,g), n>2, is called Ricci Riemannian manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(LX, Y) = \frac{r}{n-1}S(X, Y). \qquad \ldots(3.5)$$

An example showing the existance of Ricci Riemannian manifold was given by constructing (M^3, g) whose metric[8] in local coordinates (x^1, x^2, x^3) is as follows:

$$ds^2 = e^{x^1+x^2}(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2. \qquad \ldots(3.6)$$

Here an existance of $(AGSPRS)_n$ is shown by using the condition (3.5) on $(AGPRS)_n$ as follows:.

From (1.2) and (3.5) and since in a Ricci Riemannian manifold the scalar curvature[8] r ≠ 0, we have

$$(\nabla_X S)(Y,Z) = [\bar{A}(X) + \bar{B}(X)]S(LY,Z) + \bar{C}(Y)S(LZ,X) + \bar{D}(Z)S(LX,Y), \qquad \ldots(3.7)$$

where $\bar{A}$(X) = g(X, $\frac{n-1}{r}\alpha$), $\bar{B}$(X) = g(X, $\frac{n-1}{r}\beta$), $\bar{C}$(X) = g(X, $\frac{n-1}{r}\gamma$) and $\bar{D}$(X) = (X, $\frac{n-1}{r}\delta$), which is a $(AGSPRS)_n$ whose associated vector fields are parallel to the associated vector fields of $(AGPRS)_n$. This gives the existance of $(AGSPRS)_n$. Conversely, let us supose that $(AGSPRS)_n$ satisfies the condition (3.5).

From (1.7) and (3.5) , we have

$(\nabla_X S)(Y, Z) = [\tilde{A}(X) + \tilde{B}(X)]\, S(Y, Z) + \tilde{C}(Y)S(X, Z)$

$+ \widetilde{D}(Z)S(Y, X),$ …(3.8)

where $\tilde{A}(X) = g(X, \frac{r}{n-1}\alpha)$, $\tilde{B}(X) = g(X, \frac{r}{n-1}\beta)$, $\tilde{C}(X) = g(X, \frac{r}{n-1}\gamma)$ and $\widetilde{D}(X) = g(X, \frac{r}{n-1}\delta)$,

which is a $(AGPRS)_n$ whose associated vector fields are parallel to the the associated vector fields of $(AGSPRS)_n$. Thus we have the following theorem

Theorem 3.4: A $(AGPRS)_n$ is a $(AGSPRS)_n$ with parallel associated vector fields if and only if both the manifolds are Ricci Riemannian.

4.The length of the Ricci tensor and the scalar curvature of $(AGSPRS)_n$

In this section we shall obtain the length of the Ricci tensor and investigate the nature of scalar curvature of $(AGSPRS)_n$.

Putting Y = Z = e_i in (2.4) and i is summed for $1 \leq i \leq n$, we get

$dr(X) = 2[A(X) + B(X) - D(X)]\, |S|^2 - 2S(LX, \alpha) -$

$2S(LX, \beta) + 2S(LX, \delta)$ …(4.1)

From (2.3) and (4.1) we get

$[A(X) + B(X) - 2D(X)]\, |S|^2 - 2S(LX, \alpha) - 2\, S(LX, \beta) -$

$S(LX, \gamma) + S(LX, \delta) = 0.$ …(4.2)

From (4.2) if A(X) + B(X) ≠ 2D(X) we can write

$$|S|^2 = \frac{2S(LX,\alpha) + 2\,S(LX,\beta) + S(LX,\gamma) - S(LX,\delta)}{A(X) + B(X) - 2D(X)} \qquad \ldots (4.3)$$

Hence we can state the following theorem:

Theorem 4.1: In a $(AGSPRS)_n$ the following relation holds

$$[A(X) + B(X) - 2D(X)]\,|S|^2 - 2S(LX,\alpha) - 2\,S(LX,\beta) - S(LX,\gamma) + S(LX,\delta) = 0.$$

and if A(X) + B(X) ≠ 2D(X), then the length of the Ricci tensor S is given by

$$|S|^2 = \frac{2S(LX,\alpha) + 2\,S(LX,\beta) + S(LX,\gamma) - S(LX,\delta)}{A(X) + B(X) - 2D(X)}$$

Note1: If $A = B = C = D$, then from (4.2) we get S(LX, α) = 0.

This result was proved by the present author for special pseudo Ricci symmetric manifold in his paper[5].

NOTE 2: If A = C = D, then from (4.2) we get

$$|S|^2 = \frac{2S(LX,\alpha) + 2\,S(LX,\beta)}{B(X) - A(X)}\,, A(X) \neq B(X)$$

This result was proved by the present author for almost special pseudo Ricci symmetric manifold in his paper[6].

NOTE 3: If A = B, then from (4.2) we get

$$|S|^2 = \frac{4S(LX,\alpha) + S(LX,\gamma) - S(LX,\delta)}{2[A(X) - D(X)]}\,, A(X) \neq D(X).$$

This result was proved by the present author for generalized special pseudo Ricci symmetric manifold in his paper[7].

Again we suppose that the scalar curvature is constant i.e.

$$dr(X) = 0. \qquad \ldots (4.4)$$

From (2.3) and (4.4) we get

$$|S|^2 = -\frac{S(LX,\gamma) + S(LX,\delta)}{A(X) + B(X)} \qquad \ldots (4.5)$$

Again if (4.5) holds in $(AGSPRS)_n$, then from (2.3) we get dr(X) = 0 i.e. the scalar curvature is constant . This leads to the following theorem:

Theorem 4.2: The scalar curvature of $(AGSPRS)_n$ is constant if and only if $|S|^2 = -\dfrac{S(LX,\gamma) + S(LX,\ \delta)}{A(X) + B(X)}$

Again if the scalar curvature is constant , we get from (4.1),

$$[A(X) + B(X) - D(X)]|S|^2 - S(LX,\alpha) - S(LX,\beta) + S(LX,\delta) = 0 \qquad \text{...(4.6)}$$

From (1.8), (1.9), (1.11) and (4.6) we get

$$S(LX,\xi) = g(X,\xi)\,|S|^2 \qquad \text{...(4.7)}$$

where

$$\xi = \alpha + \beta - \delta \qquad \text{...(4.8)}$$

From (1.4) and (4.7) we get

$$L^2\xi = |S|^2\,\xi, \text{ for all X.} \qquad \text{...(4.9)}$$

Let κ be the eigen value of the Ricci tensor L of type (1,1) corresponding to the eigen vector ξ, then from (4.9) we get,

$$\kappa = \pm|S|. \qquad \text{...(4.10)}$$

Hence the Ricci tensor L of type (1,1) has two eigen values, namely, + $|S|$ and - $|S|$ corresponding to the eigenvector ξ. This leads to the following theorem:

Theorem 4.3: In a $(AGSPRS)_n$, $(n > 2)$, with constant scalar curvature, the Ricci tensor L of type (1,1) has two eigen values, namely, + $|S|$ and - $|S|$ corresponding to the eigen vector $\alpha + \beta - \delta$ where α, β and δ are the associated vector fields of the manifold and $|S|$ is the length of the Ricci tensor S of type (0, 2).

Now let us suppose that $+|S|$ has multiplicity m and -$|S|$ has multiplicity m – n, then r = (2m - n) $|S|$. …(4.11)

If the scalar curvature is zero, then from (4.11) we get, n = 2m i.e. the manifold is even dimensional.

Hence we can state the following theorem:

Theorem 4.4: A $(AGSPRS)_n$, $(n > 2)$, with zero scalar curvature is even dimensional.

Again interchanging Y and Z in (1.7) we get

$(\nabla_X S)(Z,Y) = [A(X) + B(X)]\, S(LZ,Y) + C(Z)S(LY,X) + D(Y)S(LX,Z)$. …(4.12)

Since $(\nabla_X S)(Y, Z) = (\nabla_X S)(Z, Y)$ we get from (1.7) and (4.12)

$E(Z)S(LX,Y) = E(Y)S(LX,Z)$, …(4.13)

where,

$E(X) = C(X) - D(X)$. …(4.14)

Let

E(X) = g(X, η), where $\eta = \gamma - \delta$. …(4.15)

Putting X = Y = $e_i$in (4.13) and i is summed for $1 \leq i \leq n$, we get

E(Z)$|S|^2$ = S(LZ, η). …(4.16)

Putting Z = η in (4.13) and using (4.16) we get

$S(LX,Y) = |S|^2\, T(X)T(Y)$, …(4.17)

where

T(X) = g(X, ζ) , $\zeta = \frac{\eta}{\sqrt{E(\eta)}}$ is a unit vector field. …(4.18)

Thus we have the following theorem:

Theorem 4.5: In a $(AGSPRS)_n$, $(n > 2)$, the Ricci tensor has the form $S(LX,Y) = |S|^2\, T(X)T(Y)$, where $T(X) = g(X,\zeta)$, $\zeta = \frac{\gamma-\delta}{\sqrt{g(\gamma-\delta,\gamma-\delta)}}$ is a unit vector field, γ and δ are associated vector fields of the manifold and $|S|$ is the length of the Ricci tensor S of type (0, 2).

5. $(AGSPRS)_n$ with cyclic parallel Ricci tensor

In this section we consider the effect of cyclic parallel Ricci tensor in a $(AGSPRS)_n$.

The Ricci tensor S of type (0, 2) is called cyclic parallel [Gray[10]] if it satisfies the condition

$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$ …(5.1)

From (1.7) and (5.1) we get

P(X) S(LY, Z) + P(Y)S(LZ, X) + P(Z)S(LX, Y) = 0, …(5.2)

where

$P(X) = A(X) + B(X) + C(X) + D(X).$ …(5.3)

Now we have the Walker's lemma [Walker[11]]:

Walker's Lemma: If a_{ij} , b_i are numbers, $a_{ij} = a_{ji}$ and $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$, for i, j, k = 1, 2, 3, …, n, then either all a_{ij} are zero or, all b_i are zero. In local coordinates the 1-form P can be written as P_i and the tensor S in the form S(LX, Y) can be written as $R_{it}R_j^t = \bar{R}_{ij}$. In local coordinates the equation (5.2) can be expressed as

$\bar{R}_{ij}P_k + \bar{R}_{jk}P_i + \bar{R}_{ki}P_j = 0.$ …(5.4)

Hence by Walker's Lemma we get either $P_i = 0$ or, $\bar{R}_{jk} = 0$

i.e. either P(X) = 0 or, S(LY, Z) = 0.

Since S(LY, Z) ≠ 0, by definition of $(AGSPRS)_n$, P(X) = 0. Hence from (5.3) we get

A(X) + B(X) + C(X) + D(X) = 0. …(5.5)

Conversely, we suppose that the relation (5.5) holds in a $(AGSPRS)_n$, then from (1.7) and (5.5) we get

$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$

i.e., the Ricci tensor is cyclic parallel. Hence we can state the following theorem:

Theorem 5.1: In a $(AGSPRS)_n$ the Ricci tensor S of type (0, 2) is cyclic parallel if and only if

A(X) + B(X) + C(X) + D(x) = 0.

6. Ricci curvature of $(AGSPRS)_n$

In this section we shall find a property of Ricci curvature of $(AGSPRS)_n$. If we replace Y and Z by X in (1.7) we get

$(\nabla_X S)(X, X) = 0\ [A(X) + B(X) + C(X) + D(X)]S(LX, X).$ …(6.1)

Since the Ricci tensor is non-zero, from (6.1) we get,

$\nabla_X S)(X, X) = 0$ if and only if A(X) + B(X) + C(X) + D(x) = 0.

Thus we have the following theorem:

Theorem 6.1: In a $(AGSPRS)_n$ Ricci curvature in the direction of any vector field is covariantly constant if and only if A(X) + B(X) + C(X) + D(X) = 0.

Note: Every Riemannian manifold admitting cyclic parallel Ricci tensor has covariantly constant Ricci curvature in any direction of a vector field but every manifold having covariantly constant Ricci curvature in any direction of a vector field may not admit cyclic parallel Ricci tensor. Here from Theorem 5.1 and Theorem 6.1 we find that the condition $A(X) + B(X) + C(X) + D(x) = 0$ makes it possible.

7. Conformally flat Ricci semi symmetric $(AGSPRS)_n$ *, (n>3)*

In this section we shall find the nature of $(AGSPRS)_n$ if it is a conformally flat Ricci semi symmetric manifold.

In a conformally flat Riemannian manifold (M^n,g), n>3, the Riemannian curvature tensor of type (1,3) [Eisenhart[12]] R is given by

$R(X,Y)Z = \frac{1}{n-2}[g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X - S(X,Z)Y] - \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].$ …(7.1)

Now we suppose that this manifold is a Ricci semi symmetric [Szabo[13]] manifold, then we have

$R(X, Y)\cdot S = 0.$ …(7.2)

From (7.1) and (7.2) we get

$g(Y,Z)S(LX,W) - g(X,Z)S(LY,W) + g(Y,W)S(LX,Z) - g(X,W)S(LY, Z) - \frac{r}{n-1}[g(Y, Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) - g(X,W)S(Y,Z)] = 0.$ …(7.3)

Putting $Y = Z = e_i$ in (7.3) where $\{e_i\}$, i = 1, 2, 3, ….,n is an orthonormal basis of the tangent space at each point, i is summed for $1 \leq i \leq n$, we get

$n[S(LX, W) - \frac{r}{n-1}S(X, W)] - [|S|^2 - \frac{r^2}{n-1}]g(X, W) = 0.$ …(7.4)

From (4.17) and (7.4) we get

$S(X, W) = a\, g(X, W) + b\, T(X)T(W),$ …(7.5)

where $a = \frac{r^2 - (n-1)|S|^2}{nr}$, $b = \frac{(n-1)|S|^2}{r}$, if $r \neq 0$.

The definition of quasi Einstein manifold[14] as follows:

A non-flat Riemannian manifold (M^n, g), (n>2) is called quasi Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$S(X,Y) = pg(X,Y) + qH(X)H(Y),$ (7.6)

where p, q are scalars $q \neq 0$ and H is a non-zero 1-form defined by H(X) = g(X, μ), μ is a unit vector field. Many works[15,16], on this manifold and its generalizations have been done by several authors.

Here we get from (7.5), a and b are scalars, since $|S| \neq 0$ implies b $\neq$ 0. Again from (4.18) we have T(X) = g(X, ζ), where ζ is a unit vector field. Thus (7.5) satisfies all the conditions of quasi Einstein manifold. Therefore it follows that a conformally flat Ricci semi symmetric $(AGSPRS)_n$ $(n > 3)$ with non-zero scalar curvature is a quasi Einstein manifold introduced by Chaki and Maity in 2000. Thus we can state the following theorem:

Theorem 7.1: A conformally flat Ricci semi symmetric $(AGSPRS)_n$ $(n > 3)$ with non-zero scalar curvature is a quasi Einstein manifold.

8. Conclusion

After preliminaries an existence theorem of $(AGSPRS)_n$ has been proved. It is shown that a $(AGPRS)_n$ is a $(AGSPRS)_n$ with the same associated 1-forms if and only if both the manifolds satisfy the condition L^2X = LX. In the next section it is shown that in a $(AGSPRS)_n$, $(n > 2)$, with constant scalar curvature, the Ricci tensor L of type (1,1) has two eigen values, namely, + $|S|$ and - $|S|$ corresponding to the eigen vector $\alpha + \beta - \delta$ where α, β and δ are the associated vector field of the manifold and $|S|$ is the length of the Ricci tensor S of type (0, 2) and a $(AGSPRS)_n$, $(n > 2)$, with zero scalar curvature is even dimensional. In section 3, it is shown that in a $(AGSPRS)_n$, $(n > 2)$, the Ricci tensor has the form $S(LX, Y) = |S|^2\, T(X)T(Y)$, where $T(X) = g(X, \zeta)$, $\zeta = \frac{\beta-\gamma}{\sqrt{g(\beta-\gamma,\beta-\gamma)}}$ is a unit vector field, β and γ are associated vector field of the manifold. In section 4, it is shown that in a $(AGSPRS)_n$ the Ricci tensor S of type (0, 2) is cyclic parallel if and only if $A(X) + B(X) + C(X) + D(X) = 0$ and In section 5 it is shown that in a $(AGSPRS)_n$ Ricci curvature in the direction of any vector field is covariantly constant if and only if $A(X) + B(X) + C(X) + D(X) = 0$. Every Riemannian manifold admitting cyclic parallel Ricci tensor has covariantly constant Ricci curvature in any direction of a vector field but every manifold having covariantly constant Ricci curvature in any direction of a vector field does not admit cyclic parallel Ricci tensor. Here we find that the condition $A(X) + B(X) + C(X) + D(X) = 0$ makes it possible.

Finally, it is shown that a conformally flat Ricci semi symmetric $(AGSPRS)_n$, $(n > 3)$, with non-zero scalar curvature is a quasi Einstein manifold.

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Three Particle Short Range Azimuthal Correlation of Pions in ^{32}S -AgBr interactions at 200 AGeV

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[**Abstract:** This paper presents a study of many particles short range azimuthal correlation of pions emitted in ^{32}S -AgBr interactions at 200 AGeV by dividing the total data set into three sub-sets having different range of multiplicities. The experimental results are compared with those of Monte Carlo simulated events to look for true dynamical correlation. The data show strong many particle correlation among the produced particles for all the multiplicity sub-sets. No significant dependence of correlation on multiplicity has been observed].

Keywords: *Heavy ion interactions, produced particles, three-particle azimuthal correlation.*

1. Introduction

Studies of nuclear matter under extremes of energy and density have become a subject of increasing interest because of the possibility of

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observing some exotic phenomena. The study of correlations among the produced particles presents significant features of the nuclear interaction and is a potential source of information. These correlations can provide direct information about the later stage of the reaction when nuclear matter is highly excited and diffused[1]. During a period of more than two decades, several studies using well-known two–particle and three–particle correlation have been reported in different types of interactions at various energies[2–25].

The particles produced in high energy interactions (like hadron-hadron, hadron-nucleus) seem to prefer to be emitted in a correlated fashion. But it is not possible to say with certainty why they prefer to do so. The existing explanations are not at all conclusive. Some people thought that the larger part of the correlation effects observed is conditioned by the production of the well-known resonances, hot multi-nucleon fireballs or formation of the exotic state of nuclear matter, the quark-gluon-plasma. Others found the experimental data to favour the formation of heavier intermediate states, clusterisation or the so-called side splash phenomenon etc. A detailed correlation study is essential to look for the exact reason of the correlated emission of the particles speculated by different theorists.

During recent years only a few works have been reported where two and three particle correlations have been studied in heavy ion interactions at SPS energies[13]. We have made an investigation on the three particle azimuthal correlations for the produced particles in ^{32}S - AgBr interaction at 200 AGeV. The experimental data on three particle correlation function for the showers is compared with those obtained by Monte-Carlo simulation (assuming an independent emission model) to search for true dynamical effect. Any significant excess of the experimental data over Monte Carlo values are termed as dynamical surplus. The dynamical surplus may signify the presence of dynamical correlation among the particles.

We have studied three particle azimuthal correlation among pions produced in ^{32}S -AgBr interaction at 200 AGeV . Moreover, we have investigated the dependence of correlation coefficient on multiplicity by dividing the total data set into different sub sets having different range of multiplicities.

2. *Experimental details*

Stacks of Ilford G5 nuclear emulsion plates exposed to ^{32}S beam of energy 200 AGeV at CERN are used in this work. The scanning of the plates is performed with the help of the LeitzMetaloplan microscope with a 10X objective and 10X ocular lens provided with a semi-automatic scanning stage. Each plate was scanned by two independent observers to increase the scanning efficiency. Each of the selected events has been examined under an oil-immersion 100X objective. The resolution of the coordinate measurements is 1μm along the X and Y axes and 0.5 μm along the Z axis. Here the nuclear emulsion serves the purpose of target as well as detector. In our detector the uncertainty in the azimuthal angle is estimated to be 5^0. Such a high resolution makes the emulsion a suitable detector for this kind of short-range correlation study.

According to the usual emulsion methodology [18], relativistic charged particles with ionization $I \leq 1.4 I_0$ (I_0 being the minimum ionization) and velocity ≥ 0.7c, are termed as shower tracks. They are mostly pions.

For our present analysis, we have taken into consideration these shower tracks for azimuthal correlation. We have chosen 350 events of ^{32}S-AgBr interactions at 200 AGeV. The azimuthal angle (ϕ) were measured for each tracks by taking the coordinates of the interaction point (X_0 ,Y_0 ,Z_0), coordinates (X_1 ,Y_1 ,Z_1) at the end of the linear portion of each secondary track and coordinate (X_i ,Y_i ,Z_i) of a point on the incident beam. For our present analysis we have used the variable ϕ .

It is worthwhile to mention that emulsion technique possesses very high spatial resolution, which makes them very effective detector for studying the three particle correlation phenomena.

3. *Method of Analysis*

Three particle inclusive correlation functions for phase space variable z can be defined[16] as

$R(z_1, z_2, z_3) =$

$$\frac{\rho_3(z_1,z_2,z_3)+2\rho_1(z_1)\rho_1(z_2)\rho_1(z_3)}{\rho_1(z_1)\rho_1(z_2)\rho_1(z_3)}-\frac{\rho_2(z_1,z_2)\rho_1(z_3)+\rho_2(z_2,z_3)\rho_1(z_1)+\rho_2(z_3,z_1)\rho_1(z_2)}{\rho_1(z_1)\rho_1(z_2)\rho_1(z_3)}$$

…(1)

where the quantities

$$\rho_1(z)=\frac{1}{\sigma}\frac{d\sigma}{dz};\rho_2(z_1,z_2)=\frac{1}{\sigma}\frac{d^2\sigma}{dz_1dz_2};\quad \rho_3(z_1,z_2,z_3)=\frac{1}{\sigma}\frac{d^3\sigma}{dz_1dz_2dz_3};$$

represent one, two, and three particle densities respectively.

In terms of number of particles, R can be represented as

$$\begin{aligned}R(z_1,z_2,z_3) = {} & N_T^2 N_3(z_1,z_2,z_3)/[N_1(z_1)N_1(z_2)N_1(z_3)] \\ & - N_T N_2(z_1,z_2)/[N_1(z_1)N_1(z_2)] \\ & - N_T N_2(z_2,z_3)/[N_1(z_2)N_1(z_3)] \\ & - N_T N_2(z_3,z_1)/[N_1(z_3)N_1(z_1)] + 2\end{aligned} \qquad ...(2)$$

where N_T is the total number of inelastic events, $N_2(z_1, z_2)$ is the number of pairs of particles having one particle between phase space intervals z_1 to z_1+dz_1 and other particle between the phase space interval z_2 to z_2+dz_2 and $N_3(z_1, z_2, z_3)$ is the number of triplets of particles having one particle between the interval z_1 to z_1+dz_1 other particle between z_2 to z_2+dz_2 and the third particle between z_3 to z_3+dz_3 in an event .

For the purpose of study of **azimuthal correlation** among shower particles, ϕ is chosen as phase space variable (ϕ being the azimuthal angle of emission of the particles).

In terms of ϕ,

$$R(\phi_1,\phi_2,\phi_3) = N_T^2 N_3(\phi_1, \phi_2, \phi_3)/[N_1(\phi_1)N_1(\phi_2) N_1(\phi_3)] - N_T N_2(\phi_1,\phi_2)/[N_1(\phi_1)N_1(\phi_2)] - N_T N_2(\phi_2, \phi_3)/[N_1(\phi_2)N_1(\phi_3)] - N_T N_2(\phi_3,\phi_1)/[N_1(\phi_3)N_1(\phi_1)] + 2 \qquad ...(3)$$

4. Monte Carlo Simulation

Correlation between the particles produced in high energy heavy ion collisions can be studied by observing azimuthal correlation (ϕ) among them. Apart from the presence of any true dynamics, correlation may arise due to the following reasons:

(a) The broad multiplicity distribution of produced particles.

(b) The dependence of single particle spectrum $\frac{1}{\sigma}\frac{d\sigma}{dz}$ on the multiplicity.

(c) Trivial statistical fluctuations.

We have compared the experimental data with the data obtained by Monte Carlo Simulation assuming Independent Emission Model (IEM), to search for the non-trivial dynamical correlation among the produced particles in ^{32}S - AgBr interactions. The simulation is made using the following assumptions:

(a) The produced particles are emitted statistically independently.

(b) The multiplicity distribution of the Monte Carlo events is the same as the empirical multiplicity spectrum of the real ensemble.

(c) The single particle spectrum $\frac{1}{\sigma}\frac{d\sigma}{dz}$ of the simulated events reproduces the empirical multiplicity distribution $\frac{1}{\sigma}\frac{d\sigma}{dz}$ of the real ensemble.

This method has been successfully applied for hadron-nucleus and nucleus-nucleus interactions[2, 4–6].

It may be concluded that if one finds any excess in experimental values over the Monte Carlo simulated values, then there may be some kinematical reason within the reaction process which may leads towards short range dynamical correlation among produced particles. We will denote the experimental normalized correlation function by R and that of the Monte Carlo calculated events by R_M. The difference between experimental and Monte Carlo values (R_D = R-R_M) is called as dynamical surplus and may be interpreted as a measure of genuine dynamical correlation.

5. Experimental Results and Discussions

For studying the multiplicity dependence of correlation, we have divided the whole data set into three sub-sets as following:

Set-I : Number of pions< 70

Set-II : Number of pions are in between (70 – 110)

Set-III : Number of pions> 110

Sub-sets have more or less equal number of events.

Normalised three-particle azimuthal correlation function R has been calculated for different values of azimuthal angle (ϕ) for all the data sets using eqn.(3) . Three particle short range correlation has been investigated by plotting the values of correlation function R(ϕ_1, $\phi_2 = \phi_1$, $\phi_3 = \phi_1$) i.e. the diagonal elements of the correlation matrix R ($\phi_{1,}\phi_{2,}\phi_3$) against ϕ variable.

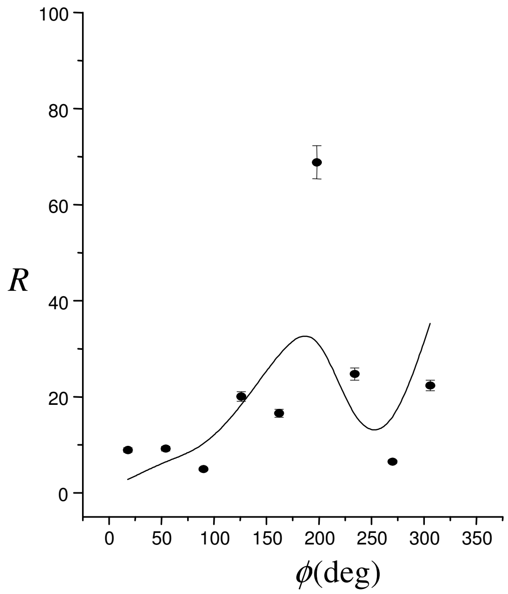

Fig. 1(a)

Three particle correlation function R for different values of ϕ for data set I. The solid curve represents Monte Carlo simulated values.

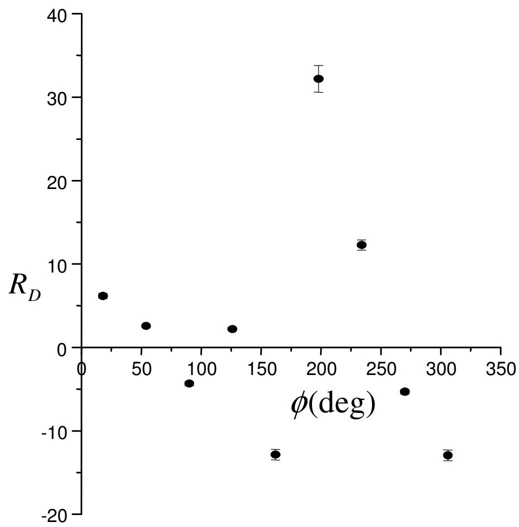

Fig.1(b)

Dynamical surplus values R_D of three particle correlation for different values of ϕ for data set I.

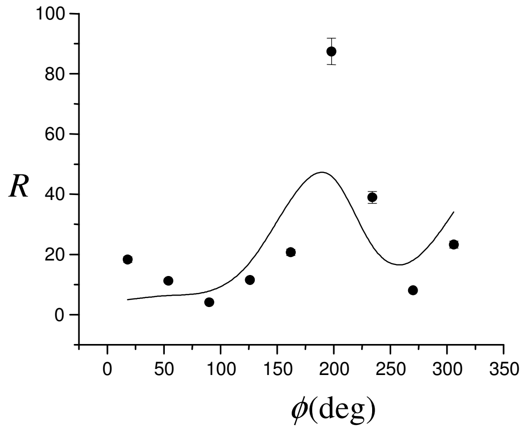

Fig.2(a)

Three particle correlation function R for different values of ϕ for data set II. The solid curve represents Monte Carlo simulated values.

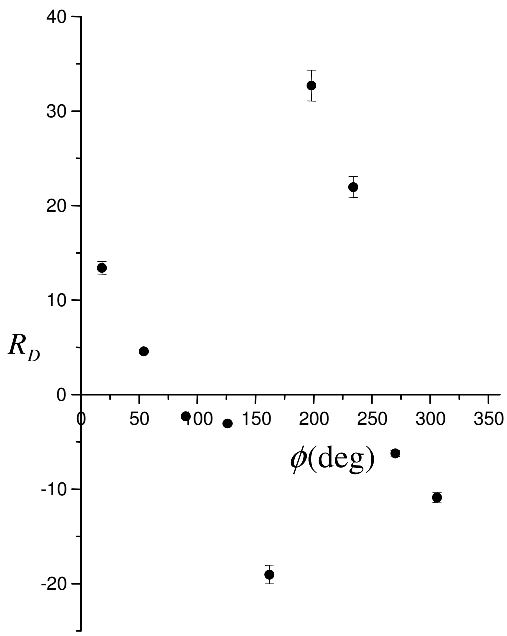

Fig. 2(b)

Dynamical surplus values R_D of three particle correlation for different values of ϕ for data set II.

Fig.1(a) presents the plot of R vs. ϕ corresponding to three-particle azimuthal correlation among produced particles for data set I. The solid curve in this figure represents the values of correlation function due to Monte Carlo simulation and the dots indicate the experimental values. The errors shown are statistical in origin. Similar things for data set II and III have been shown in Fig. 2(a) and 3(a) respectively. The dynamical surplus correlation R_D for different values of ϕ are shown in Fig.1(b), 2(b) and 3(b) for data sets I, II and III respectively. For all multiplicity ranges three particle short range correlation exists over the entire azimuthal angle space.

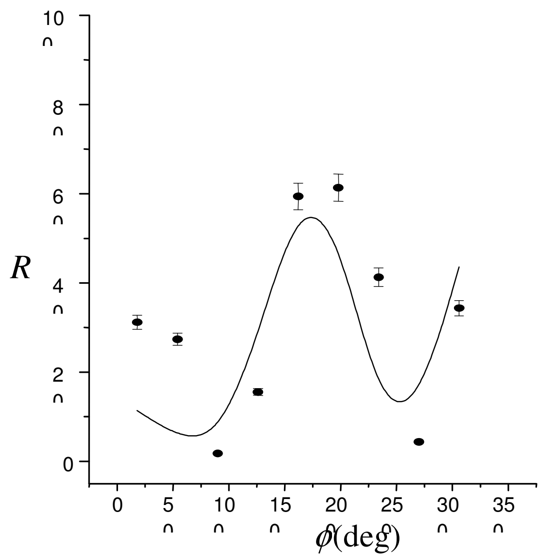

Fig.3(a)

Three particle correlation function R for different values of ϕ for data set III. The solid curve represents Monte Carlo simulated values.

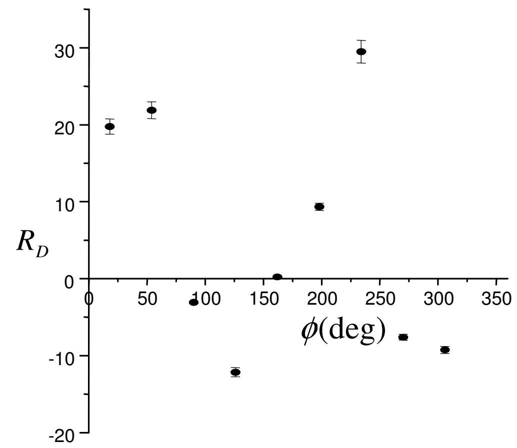

Fig. 3(b)

Dynamical surplus values R_D of three particle correlation for different values of ϕ for data set III.

It is observed that the correlation effect is extremely significant at azimuthal angle ϕ around 198^0 for the data set I and prominent correlation exists at around $\phi = 162^0$, 234^0 and 306^0 .

For data set II, it is seen that the correlation is most prominent at values of ϕ around 198^0 and the three particle correlation is prominent around $\phi = 18^0$, 162^0 and 234^0 .

For data set III, it is observed that the maximum correlation exists at a value of ϕ around $234^0$and prominent correlation exists at around $\phi = 18^0$, 54^0 and 126^0 .

6. Conclusion

It may be concluded that strong three particle azimuthal correlation is indicated by all the three sub-sets of different range of multiplicities. No significant dependence of correlation on multiplicity has been observed.

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